A STATE SPACE APPROACH TO CONTROL OF INTERCONNECTED SYSTEMS*

RAFFAELLO D'ANDREA†, CEDRIC LANGBORT†, AND RAMU CHANDRA†

Abstract. We present a state space approach to controlling systems with a highly structured interconnection topology. It is shown that by capturing these systems as fractional transformations on temporal and spatial operators, many standard results in control – such as the bounded real lemma, H-infinity optimization, and robustness analysis – can be generalized accordingly. The state space formulation yields conditions that can be expressed as linear matrix inequalities.

Key words. Distributed control, H-infinity, interconnected systems, linear matrix inequalities.

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1. Introduction. Many systems consist of similar units which directly interact with their nearest neighbors. Even when these units have tractable models and interact with their neighbors in a simple and predictable fashion, the resulting system often displays rich and complex behavior when viewed as a whole. There are many examples of such systems, including automated highway systems [42], airplane formation flight [48, 7], satellite constellations [43], cross-directional control in paper processing applications [44], and very recently, micro-cantilever array control for massively parallel data storage [37]. One can also consider lumped approximations of partial differential equations (PDEs) – examples include the deflection of beams, plates, and membranes, and the temperature distribution of thermally conductive materials [46].

An important aspect of many of these systems is that sensing and actuation capabilities exist at every unit. In the examples above, this is clearly the case for automated highway systems, airplane formation flight, satellite constellations, and cross-directional control systems. With the rapid advances in micro electro-mechanical actuators and sensors, however, we will soon be able to instrument systems governed by partial differential equations with distributed arrays of actuators and sensors, rendering lumped approximations with collocated sensors and actuators valid mathematical abstractions.

If one attempts to control these systems using standard control design techniques, severe limitations will quickly be encountered as most optimal control techniques cannot handle systems of very high dimension *and* with a large number of inputs and outputs. It is also not feasible to control these systems with centralized schemes – the typical outcome of most optimal

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control design techniques – as these require high levels of connectivity, impose a substantial computational burden, and are typically more sensitive to failures and modeling errors than decentralized schemes.

In order for any optimal control technique to be successful, the structure of the system must be exploited in order to obtain tractable algorithms. In this paper, we present a state space approach to controlling systems with a highly structured interconnection topology; in particular, we consider linear, spatially invariant systems that can be captured as fractional transformations on temporal and spatial operators. By doing so, many standard results in control – such as the bounded real lemma, H-infinity optimization, and robustness analysis – can be generalized accordingly. The state space formulation yields conditions that can be expressed as linear matrix inequalities (LMIs) [3], resulting in tractable computational tools for control design and analysis.

The types of problems considered in this paper have a long history. In [35], optimal regulation for a countably infinite number of objects is considered by employing a bilateral Z-transform, which is analogous to the spatial shift operators introduced in this paper. In [5] it was shown that discretization of certain classes of PDEs result in control systems defined on modules, and that the resulting structure can be exploited to reduce computational effort.

Recently [1], control problems for spatially invariant systems with quadratic performance criteria (such as $\mathcal{H}_2$ and $\mathcal{H}_\infty$) are tackled by extending familiar frequency-domain concepts for one-dimensional systems. The control design problem is then solved for a parameterized (over frequency) system of finite-dimensional systems. It is also shown that the optimal controller has a degree of spatial localization (similar to the plant) and can therefore be implemented in a distributed fashion.

Robust stability analysis problems for multidimensional systems are considered in [26]. Results are derived using Laplace transforms in several complex variables which show that the problem can be solved by the methods of structured uncertainty analysis (μ analysis) [39].

An important and practical application, that of cross-directional control of paper machine processes, is considered in [44]. The notion of loop shaping [32] is extended to two-dimensional systems (one temporal, one spatial). The special structure of the paper machine problem, and of similar problems, is exploited to apply the results in [1] and obtain a computationally attractive practical control design methodology to address performance and robustness issues.

This paper is based in part on the work in [9, 15, 19, 10, 13, 11, 12, 17, 16, 6, 29], and is organized as follows. In Section 2 we introduce the systems considered in this paper. In Section 3 we present linear matrix inequality conditions for analysis – determining whether an infinite extent interconnected system is well-posed, stable, and contractive, followed by connections to finite extent problems in Section 4. Controller synthesis

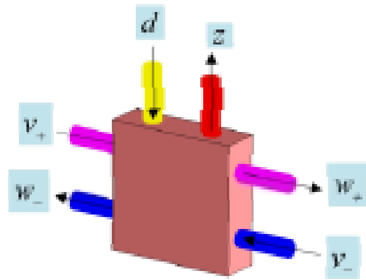

FIG. 1. *Basic building block, one spatial dimension.*

and controller implementation are discussed in Section 5, followed by a numerical example in Section 6. Extensions are discussed in Section 7.

2. Interconnected systems. Consider the diagram in Figure 1. It consists of a finite dimensional, linear time invariant system governed by the following state space equations:

$$\begin{bmatrix} \dot{x}(t) \\ w(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} A_{\mathtt{TT}} & A_{\mathtt{TS}} & B_{\mathtt{T}} \\ A_{\mathtt{ST}} & A_{\mathtt{SS}} & B_{\mathtt{S}} \\ C_{\mathtt{T}} & C_{\mathtt{S}} & D \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \\ d(t) \end{bmatrix} \tag{2.1}$$

where

$$v(t) = (v_+(t), v_-(t)), \qquad w(t) = (w_+(t), w_-(t)) \ . \tag{2.2}$$

We assume that $v_+(t)$ and $w_+(t)$ are the same size, and that $v_-(t)$ and $w_-(t)$ are the same size. We will consider various interconnections of large numbers of these subsystems. We will index these units by the integer-valued variable s – the spatial independent variable – and thus have the following equations, valid for each s in some given range, and for each $t \geq 0$:

$$\begin{bmatrix} \dot{x}(t,s) \\ w(t,s) \\ z(t,s) \end{bmatrix} = \begin{bmatrix} A_{\mathtt{TT}} & A_{\mathtt{TS}} & B_{\mathtt{T}} \\ A_{\mathtt{ST}} & A_{\mathtt{SS}} & B_{\mathtt{S}} \\ C_{\mathtt{T}} & C_{\mathtt{S}} & D \end{bmatrix} \begin{bmatrix} x(t,s) \\ v(t,s) \\ d(t,s) \end{bmatrix} . \tag{2.3}$$

A remark on notation. When referring to a signal at 1) a specific location in time and space, we will use the notation $d(t,s)$; 2) over all time, but at a specific location in space, we will use the notation $d(s)$; 3) over all space, but at a specific instant in time, we will use the notation $d(t)$; 4) over all time and space, we will use the notation d.

We will consider four types of interconnections based on these identical copies of the basic building depicted in Figure 1. These are described next.

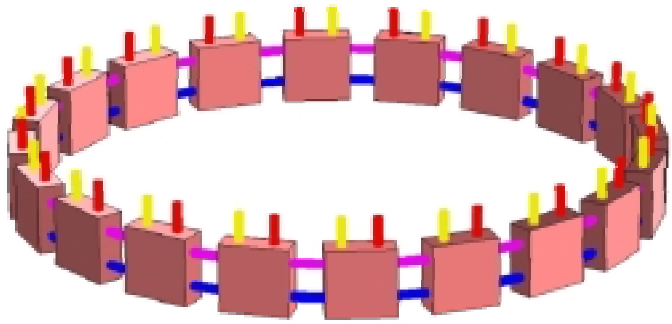

FIG. 2. *Periodic interconnection.*

2.1. Periodic interconnection. Let the number of units be N: $1 \leq s \leq N$. Define a *periodic interconnection* as follows:

$$v_+(s+1) = w_+(s), \qquad 1 \leq s \leq N-1 \tag{2.4}$$

$$v_+(s=1) = w_+(s=N) \tag{2.5}$$

$$v_-(s-1) = w_-(s), \qquad 2 \leq s \leq N \tag{2.6}$$

$$v_-(s=N) = w_-(s=1)\ . \tag{2.7}$$

This is depicted for N=20 in Figure 2. Once the interconnection has been formed, the system inputs are simply d, and the system outputs are z; v and w can be considered internal system variables.

2.2. Finite interconnection with boundary conditions. Let the number of units be N. For a given invertible matrix M, define a *finite interconnection with boundary conditions* as follows:

$$v_+(s+1) = w_+(s), \qquad 1 \leq s \leq N-1 \tag{2.8}$$

$$v_-(s-1) = w_-(s), \qquad 2 \leq s \leq N \tag{2.9}$$

$$v_+(s=1) = Mw_-(s=1) \tag{2.10}$$

$$v_-(s=N) = M^{-1}w_+(s=N)\ . \tag{2.11}$$

This is depicted for N=10 in Figure 3. As in the periodic case, once the interconnection has been formed, the system inputs are d, and the system outputs are z.

This method for imposing boundary conditions requires some explanation. It is motivated by the method of images, a tool which is often used to simplify problems governed by partial differential equations when there is an underlying symmetry inherent in the system.

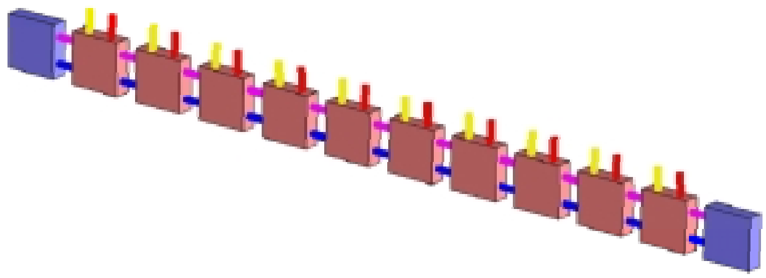

FIG. 3. *Finite interconnection with boundary conditions.*

As an illustrative example, we can revisit the central finite difference approximation of the one dimensional heat equation considered in [5]. We can express the equations governing the temperature evolution of one element as follows:

$$\dot{x}(t,s) = v_+(t,s) - x(t,s) + v_-(t,s) - x(t,s) + d(t,s) \tag{2.12}$$

$$w_+(t,s) = x(t,s) \tag{2.13}$$

$$w_-(t,s) = x(t,s) \tag{2.14}$$

$$z(t,s) = x(t,s)\ . \tag{2.15}$$

In this example, the interconnection variables are simply the nearest neighbor temperatures. The input $d(t,s)$ is the external heat flux into each subsystem, and the output $z(t,s)$ is the subsystem temperature about some equilibrium. Consider the problem when a finite number N of these elements are interconnected, and that the boundaries are insulated from their environment. This Neumann type of boundary condition can be implemented by requiring that each of the two boundary elements is in contact with an element of equal temperature. In particular, $v_+(s=1) = x(s=1) = w_-(s=1)$ and $v_-(s=N) = x(s=N) = w_+(s=N)$, which can readily be seen to correspond to $M = 1$.

A standard Dirichlet type of boundary condition can be imposed for this example by requiring that the boundary elements are in contact with an element of zero temperature; this is in fact the approach taken in [5]. A slightly modified version of this problem is to consider boundary elements that are in contact with element of opposite temperature. In particular, $v_+(s=1) = -x(s=1) = -w_-(s=1)$ and $v_-(s=N) = -x(s=N) = -w_+(s=N)$, which can readily be seen to correspond to $M = -1$. The physical interpretation of this boundary condition is that the average temperature of a boundary element and that of its virtual neighbor is zero.

These issues are further explored in [30, 29, 6].

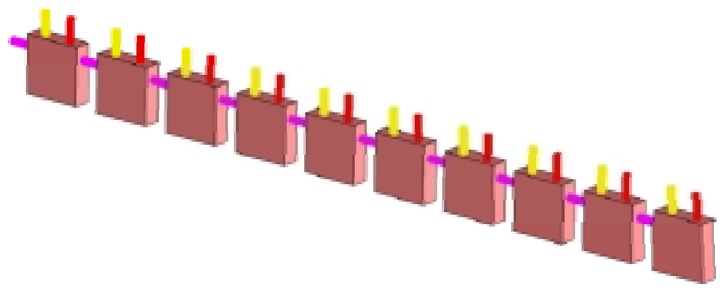

FIG. 4. *Finite interconnection with zero boundary condition.*

2.3. Finite interconnection with zero boundary condition. Let the number of units be N. Define a *finite interconnection with zero boundary condition* as follows:

$$v_+(s+1) = w_+(s), \qquad 1 \le s \le N-1 \ . \tag{2.16}$$

$$v_+(s=1) = 0 \ . \tag{2.17}$$

This is depicted for N=10 in Figure 2. Note that these types of interconnections are only defined when the interconnection variables are restricted to v_+ and w_+. In particular, a subsystem at location s can only influence a subsystem at location s^* provided that $s < s^*$. These types of interconnections are similar to the "look-ahead" systems considered in [40]. Once the interconnection has been formed, the system inputs are d, and the system outputs are z.

2.4. Infinite interconnection. Consider an infinite number of units, interconnected as follows:

$$v_+(s+1) = w_+(s) \qquad \forall s \in \mathbb{Z} \tag{2.18}$$

$$v_-(s-1) = w_-(s) \qquad \forall s \in \mathbb{Z} \ . \tag{2.19}$$

This is depicted in Figure 5. This type of interconnection is similar to the one considered by [35], where a control system is designed for an infinite number of vehicles. As was pointed out in [35], and more recently in [1], an infinite approximation may be sufficient when dealing with a large number of systems. In particular, the scale of influence of localized effects is often much less than the scale of the whole system. Even if the uncontrolled system does not satisfy this property, it is likely that the controlled system will.

There is another important reason for considering infinite extent system; as will be discussed in Section 3, if the infinite extent system is well-posed, stable, and contractive, these properties are inherited by a periodic

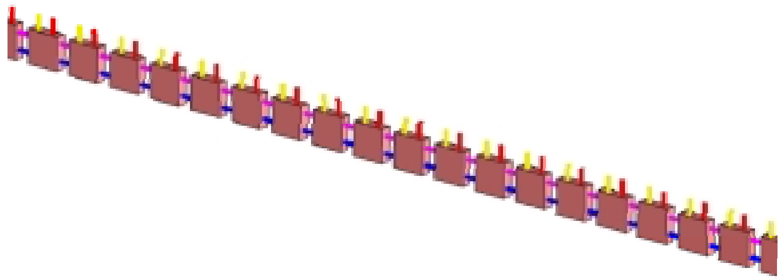

FIG. 5. *Infinite interconnection.*

interconnection, and by a finite interconnection with zero boundary conditions. When certain symmetry properties are satisfied, the finite interconnection with boundary conditions also inherits these properties.

For infinite extent systems, it is convenient to introduce the spatial shift operator $\mathbf{S}$:

$$(\mathbf{S}d)(t,s) := d(t,s+1) \; . \tag{2.20}$$

Define the following structured operator $\mathbf{\Delta_S}$:

$$\mathbf{\Delta_S} := \begin{bmatrix} \mathbf{S}\, I_+ & 0 \\ 0 & \mathbf{S}^{-1}\, I_- \end{bmatrix} \tag{2.21}$$

where $\mathbf{S}\, I_+$ denotes n_+ copies of the operator $\mathbf{S}$ along the diagonal, $\mathbf{S}^{-1}\, I_-$ denotes n_- copies of the operator $\mathbf{S}^{-1}$ along the diagonal, and n_+ and n_- are the vector dimensions of signals $w_+(t,s)$ and $w_-(t,s)$, respectively.

We may thus write the interconnected system as follows:

$$\begin{bmatrix} \dot{x}(t,s) \\ (\mathbf{\Delta_S} v)(t,s) \\ z(t,s) \end{bmatrix} = \begin{bmatrix} A_{\mathrm{TT}} & A_{\mathrm{TS}} & B_{\mathrm{T}} \\ A_{\mathrm{ST}} & A_{\mathrm{SS}} & B_{\mathrm{S}} \\ C_{\mathrm{T}} & C_{\mathrm{S}} & D \end{bmatrix} \begin{bmatrix} x(t,s) \\ v(t,s) \\ d(t,s) \end{bmatrix} \; . \tag{2.22}$$

By eliminating interconnection variables v, we can express the interconnected system as follows:

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}d(t) \tag{2.23}$$

$$z(t) = \mathbf{C}x(t) + \mathbf{D}d(t) \tag{2.24}$$

where

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} := \begin{bmatrix} A_{\mathrm{TT}} & B_{\mathrm{T}} \\ C_{\mathrm{T}} & D \end{bmatrix} + \begin{bmatrix} A_{\mathrm{TS}} \\ C_{\mathrm{S}} \end{bmatrix} (\mathbf{\Delta_S} - A_{\mathrm{SS}})^{-1} \left[\, A_{\mathrm{ST}} \;\; B_{\mathrm{S}} \,\right] \; . \tag{2.25}$$

If $(\mathbf{\Delta_S} - A_{\mathbf{SS}})^{-1}$ exists and is bounded as an operator on ℓ_2, the space of square summable sequences, operators $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ exist and are bounded, and we may readily write down the solution as:

$$x(t) = \exp(\mathbf{A}t)x(0) + \int_0^t \exp(\mathbf{A}(t-\tau))\mathbf{B}d(\tau)d\tau, \tag{2.26}$$

where $\exp(\mathbf{A}t)$ is the strongly continuous semigroup defined by

$$\exp(\mathbf{A}t) := \sum_{n=0}^{\infty} \frac{(\mathbf{A}t)^n}{n!}; \tag{2.27}$$

the reader is referred to [8, 2] for details. For the reader not familiar with semigroup theory, the key point is that the boundedness of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ allows us to formally treat these systems analogously to their finite dimensional counterparts; this should be compared with spatially continuous systems which typically have unbounded system operators [8].

We will see in Section 3 that the existence and boundedness of $(\mathbf{\Delta_S} - A_{\mathbf{SS}})^{-1}$ is equivalent to a well-posed interconnection, and is thus not a restrictive assumption.

2.5. Interconnected systems in higher dimensions. The basic building block depicted in Figure 1, and the various interconnections described in Section 2, can readily be extended to more than one spatial dimension. For example, in two dimensions, we have the following equations for the basic building block:

$$\begin{bmatrix} \dot{x}(t) \\ w(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} A_{\mathbf{TT}} & A_{\mathbf{TS}} & B_{\mathbf{T}} \\ A_{\mathbf{ST}} & A_{\mathbf{SS}} & B_{\mathbf{S}} \\ C_{\mathbf{T}} & C_{\mathbf{S}} & D \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \\ d(t) \end{bmatrix} \tag{2.28}$$

where

$$v(t) = (v_{+,1}(t), v_{-,1}(t), v_{+,2}(t), v_{-,2}(t)), \tag{2.29}$$

$$w(t) = (w_{+,1}(t), w_{-,1}(t), w_{+,2}(t), w_{-,2}(t)) \ . \tag{2.30}$$

These units can be indexed by two integer valued variables s_1 and s_2, resulting in the following equations:

$$\begin{bmatrix} \dot{x}(t,s_1,s_2) \\ w(t,s_1,s_2) \\ z(t,s_1,s_2) \end{bmatrix} = \begin{bmatrix} A_{\mathbf{TT}} & A_{\mathbf{TS}} & B_{\mathbf{T}} \\ A_{\mathbf{ST}} & A_{\mathbf{SS}} & B_{\mathbf{S}} \\ C_{\mathbf{T}} & C_{\mathbf{S}} & D \end{bmatrix} \begin{bmatrix} x(t,s_1,s_2) \\ v(t,s_1,s_2) \\ d(t,s_1,s_2) \end{bmatrix} . \tag{2.31}$$

Various interconnections can then be defined; the details are omitted. For example, a finite interconnection with boundary conditions applied on both spatial directions is depicted in Figure 6; a periodic interconnection in

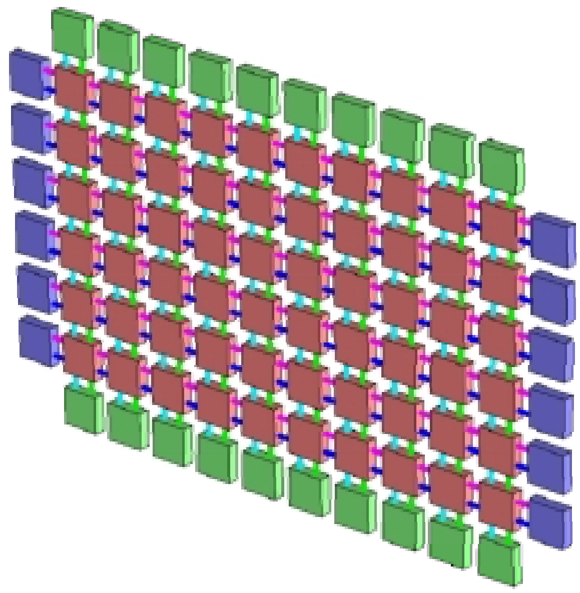

FIG. 6. *Finite interconnection with boundary conditions applied on both spatial directions.*

one spatial direction and a finite interconnection with boundary conditions on the second is depicted in Figure 7; and finally, a periodic interconnection in both spatial directions is depicted in Figure 8 (only a portion of the resulting torus is depicted in the figure for clarity). In all these figures, the inputs d and the outputs z have been omitted for clarity.

An infinite interconnection can be captured as per Equations (2.23), (2.24), and (2.25), where

$$\mathbf{\Delta_S} := \begin{bmatrix} \mathbf{S}_1\, I_{+,1} & 0 & 0 & 0 \\ 0 & \mathbf{S}_1^{-1}\, I_{-,1} & 0 & 0 \\ 0 & 0 & \mathbf{S}_2\, I_{+,2} & 0 \\ 0 & 0 & 0 & \mathbf{S}_2^{-1}\, I_{-,2} \end{bmatrix}. \tag{2.32}$$

3. Well-posedness, stability, and performance. There are three main considerations when analyzing an interconnected system: well-posedness, stability, and performance.

3.1. Well-posedness. Simply put, an interconnection is well-posed if it is physically realizable. The following simple examples illustrate the concept of well-posedness. Consider the feedback interconnection in Figure 9. Let P_1 and P_2 be unity gain systems: $w_1(t) = v_1(t), w_2(t) =$

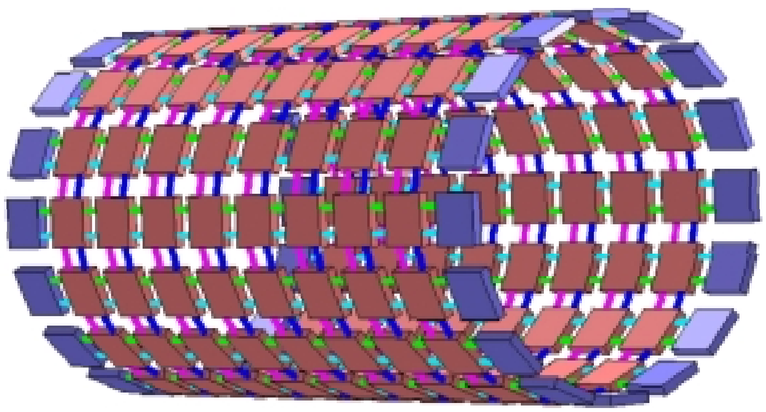

FIG. 7. *Finite interconnection with boundary conditions on one spatial direction, periodic interconnection in the second spatial direction.*

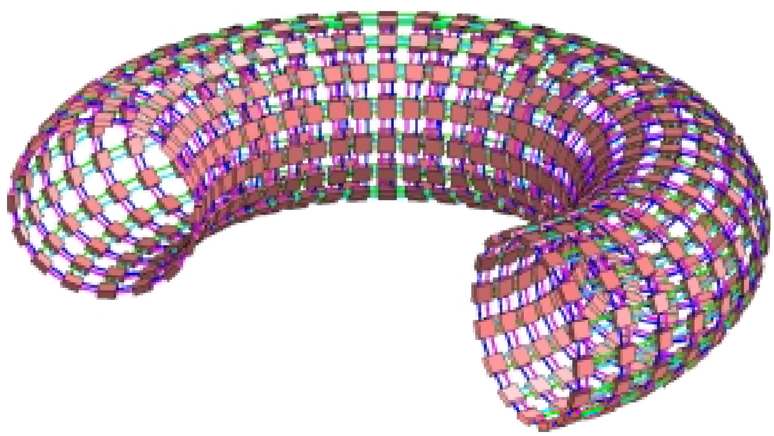

FIG. 8. *Periodic interconnection in both spatial directions.*

$v_2(t)$. This interconnection is not well-posed because there do not exist solutions to the loop equations for all possible exogenous signals n_1 and n_2.

Now let P_1 be a unity gain system, and let P_2 be a linear time invariant system with transfer function $\mathrm{P}_2(\zeta) = 1 - 1/\zeta$. This interconnection is also not well posed because the resulting transfer function from exogenous signal n_1 to interconnection signal v_1 is not proper, and in fact equal to ζ. There is thus differentiating action from one of the closed loop system inputs to one of the closed loop system outputs (all the closed loop dependent variables are considered outputs: v_1, w_1, v_2, w_2). The reader is referred to [50] for an in-depth discussion of well-posedness. We can extend the definition

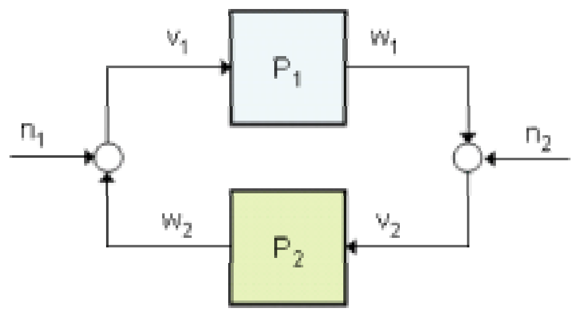

FIG. 9. *Feedback Interconnection.*

of well-posedness in [50] to the interconnections considered in this paper. In particular, we require that the transfer functions from signals injected anywhere in the loop to all the closed loop system outputs exist and are proper.

For interconnections involving a finite number of subsystems, well-posedness is equivalent to the invertibility of a matrix. In particular, for any finite interconnection, we may write the following relation between signals w, v, d, and x:

$$L_{wv} \begin{bmatrix} w(t) \\ v(t) \end{bmatrix} = L_x x(t) + L_d d(t) \tag{3.1}$$

where L_{wv}, L_x, and L_d are matrices, and L_{wv} is square. Well-posedness is then equivalent to the invertibility of matrix L_{wv}.

For infinite interconnections, well-posedness reduces to the existence and boundedness of operator $(\mathbf{\Delta_S} - A_{\mathbf{SS}})^{-1}$. As we shall see in Section 4, well-posedness of an infinite interconnection ensures well-posedness of the various types of finite interconnections considered in Section 2.

3.2. Stability. Once an interconnection has been deemed to be well-posed, we may consider system stability. We adopt the standard notion of internal exponential stability: a system is stable if in the absence of inputs d, all signals decay to zero exponentially fast – see [50, 8] for details.

3.3. Performance. If an interconnected system is well-posed and stable, we may then consider system performance. The notion of performance we adopt is that of the $\mathcal{L}_2$ gain of the system:

$$\frac{||z||}{||d||} \tag{3.2}$$

where

$$\|d\|^2 := \sum_s \int_0^\infty d^*(t,s)d(t,s)dt \ . \tag{3.3}$$

This is simply the $\mathcal{H}_\infty$ norm of the interconnected system [50, 8]. When we consider control design in Section 5, we will require that the $\mathcal{L}_2$ gain of the system be made as small as possible.

A well-posed and stable interconnection is said to be contractive if the $\mathcal{L}_2$ gain of the system is less than one. By appropriately scaling the subsystem matrices, there is no loss of generality in requiring that the interconnected system be contractive as the performance criterion.

3.4. Linear matrix inequality condition for well-posedness, stability, and performance: the infinite case. When dealing with a finite number of subsystems, stability and contractiveness can be established with the bounded real lemma (see [41], for example), which in turn can be expressed as an LMI [3]. The main reason for expressing stability and performance as an LMI is that this condition can then be exploited for controller synthesis, as was done in [38, 24, 27].

The main problem with this approach, however, is that the size of the LMI grows with the number of subsystems; computation become prohibitively expensive for even modest size problems. In addition, when these conditions are used for control design, the resulting controller is centralized, and may be difficult to implement. For example, how would one implement a centralized controller for a large number of unmanned vehicles flying in formation? We will revisit this issue in Section 5.

When dealing with an infinite number of subsystems, the direct approach described above is not feasible. We can, however, provide a sufficient (but not necessary) LMI condition for well-posedness, stability, and performance. Partition the matrices in (2.1) so as to be consistent with the partition of v and w in (2.2):

$$A_{\mathtt{SS}} =: \begin{bmatrix} A_{\mathtt{SS}_{++}} & A_{\mathtt{SS}_{+-}} \\ A_{\mathtt{SS}_{-+}} & A_{\mathtt{SS}_{--}} \end{bmatrix}, \ A_{\mathtt{ST}} =: \begin{bmatrix} A_{\mathtt{ST}_{+}} \\ A_{\mathtt{ST}_{-}} \end{bmatrix}, \ B_{\mathtt{S}} =: \begin{bmatrix} B_{\mathtt{S}_{+}} \\ B_{\mathtt{S}_{-}} \end{bmatrix}, \tag{3.4}$$

$$A_{\mathtt{TS}} =: \begin{bmatrix} A_{\mathtt{TS}_{+}} & A_{\mathtt{TS}_{-}} \end{bmatrix}, \ C_{\mathtt{S}} =: \begin{bmatrix} C_{\mathtt{S}_{+}} & C_{\mathtt{S}_{-}} \end{bmatrix} \ . \tag{3.5}$$

Define the following matrices:

$$A^+_{\mathtt{SS}} := \begin{bmatrix} A_{\mathtt{SS}_{++}} & A_{\mathtt{SS}_{+-}} \\ 0 & I \end{bmatrix}, \ A^+_{\mathtt{ST}} := \begin{bmatrix} A_{\mathtt{ST}_{+}} \\ 0 \end{bmatrix}, \ B^+_{\mathtt{S}} := \begin{bmatrix} B_{\mathtt{S}_{+}} \\ 0 \end{bmatrix}, \tag{3.6}$$

$$A^-_{\mathtt{SS}} := \begin{bmatrix} I & 0 \\ A_{\mathtt{SS}_{-+}} & A_{\mathtt{SS}_{--}} \end{bmatrix}, \ A^-_{\mathtt{ST}} := \begin{bmatrix} 0 \\ A_{\mathtt{ST}_{-}} \end{bmatrix}, \ B^-_{\mathtt{S}} := \begin{bmatrix} 0 \\ B_{\mathtt{S}_{-}} \end{bmatrix}, \tag{3.7}$$

$$A^+_{\mathtt{TS}} := \begin{bmatrix} A_{\mathtt{TS}_{+}} & 0 \end{bmatrix}, \ \ A^-_{\mathtt{TS}} := \begin{bmatrix} 0 & A_{\mathtt{TS}_{-}} \end{bmatrix}. \tag{3.8}$$

The following result is from [16]:

THEOREM 3.1. *Consider the infinite interconnected system defined by (2.23), (2.24), and (2.25). Then the interconnected system is well-posed, stable, and contractive if there exist $X_{\mathbf{T}} > 0$ and symmetric $X_{\mathbf{S}}$ such that*

$$\begin{aligned}
&\begin{bmatrix} I & 0 & 0 \\ A^-_{\mathbf{ST}} & A^-_{\mathbf{SS}} & B^-_{\mathbf{S}} \\ 0 & 0 & I \end{bmatrix}^* \begin{bmatrix} A^*_{\mathbf{TT}} X_{\mathbf{T}} + X_{\mathbf{T}} A_{\mathbf{TT}} & X_{\mathbf{T}} A^+_{\mathbf{TS}} & X_{\mathbf{T}} B_{\mathbf{T}} \\ (A^+_{\mathbf{TS}})^* X_{\mathbf{T}} & -X_{\mathbf{S}} & 0 \\ B^*_{\mathbf{T}} X_{\mathbf{T}} & 0 & -I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ A^-_{\mathbf{ST}} & A^-_{\mathbf{SS}} & B^-_{\mathbf{S}} \\ 0 & 0 & I \end{bmatrix} \\
&+ \begin{bmatrix} I & 0 & 0 \\ A^+_{\mathbf{ST}} & A^+_{\mathbf{SS}} & B^+_{\mathbf{S}} \\ C_{\mathbf{T}} & C_{\mathbf{S}} & D \end{bmatrix}^* \begin{bmatrix} 0 & X_{\mathbf{T}} A^-_{\mathbf{TS}} & 0 \\ (A^-_{\mathbf{TS}})^* X_{\mathbf{T}} & X_{\mathbf{S}} & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ A^+_{\mathbf{ST}} & A^+_{\mathbf{SS}} & B^+_{\mathbf{S}} \\ C_{\mathbf{T}} & C_{\mathbf{S}} & D \end{bmatrix} < 0 \,.
\end{aligned} \tag{3.9}$$

Note that the size of the LMI is dictated solely by the size of one subsystem, the basic building block in Figure 1.

The result generalizes to multiple spatial dimensions in a straightforward way: The matrices in (3.4) to (3.8) can be defined and partitioned accordingly, and the decision variable $X_{\mathbf{S}}$ in the LMI becomes block diagonal. For example, for two spatial dimensions $X_{\mathbf{S}}$ has the following structure:

$$X_{\mathbf{S}} = \begin{bmatrix} X_{\mathbf{S}_1} & 0 \\ 0 & X_{\mathbf{S}_2} \end{bmatrix} . \tag{3.10}$$

Note that in the absence of interconnection variables, the above reduces to the bounded real lemma.

4. Connections to finite extent problems. Since analysis of infinite extent systems can be performed in an efficient and tractable way using the tools of Section 3, it is important to know what links, if any, exist between well-posedness, stability and performance of finite and infinite interconnections. The main results can be summarized as

THEOREM 4.1. *The following hold:*

(1) A finite interconnection with zero boundary conditions is always well-posed.

(2) (a) ⇒ (b),

(3) (a) ⇒ (c) where

- *(a) The infinite interconnection is well-posed, stable and contractive.*
- *(b) For any N, the periodic interconnection with N blocks is well-posed, stable and contractive.*
- *(c) For any N, the interconnection with zero boundary conditions and N blocks is stable and contractive.*

The proof of item (2) follows from theorems in [1], once it has been realized that infinite and periodic interconnections can be seen as systems over compact groups. Item (1) is straightforward while item (3) is established by showing that the N interconnected blocks with zero boundary

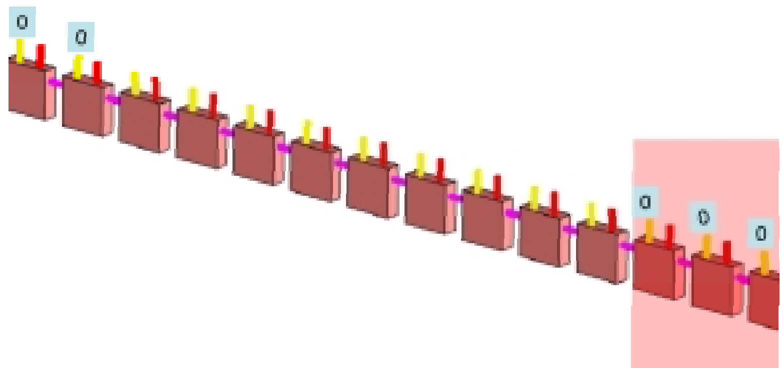

FIG. 10. *The equivalent infinite extent system for the finite interconnection with zero boundary conditions of Figure 4.*

conditions behave just as if they were embedded in an infinite extent system with a particular input d, as shown in Figure 10.

The idea of using an equivalent infinite extent system to analyze finite extent interconnections with boundary conditions is similar in nature to the "lifting techniques" used in [28, 31] and is also reminiscent of the method of images used in potential theory to simplify the domain of Laplace's equation.

Similar ideas can be used to handle the finite interconnections described in Section 2.2. A key property in this case is spatial reversibility that we shall now define.

4.1. Spatial reversibility.

DEFINITION 4.1. *Given an invertible matrix M, the basic building block defined by (2.1) and (2.2) is said to be M-reversible if there exist matrices $P = P^{-1}$, $R = R^{-1}$ and $U = U^{-1}$ such that*

$$\begin{bmatrix} P & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & U \end{bmatrix} \begin{bmatrix} A_{\mathbf{TT}} & A_{\mathbf{TS}} & B_{\mathbf{T}} \\ A_{\mathbf{ST}} & A_{\mathbf{SS}} & B_{\mathbf{S}} \\ C_{\mathbf{T}} & C_{\mathbf{S}} & D \end{bmatrix} = \begin{bmatrix} A_{\mathbf{TT}} & A_{\mathbf{TS}} & B_{\mathbf{T}} \\ A_{\mathbf{ST}} & A_{\mathbf{SS}} & B_{\mathbf{S}} \\ C_{\mathbf{T}} & C_{\mathbf{S}} & D \end{bmatrix} \begin{bmatrix} P & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & R \end{bmatrix}$$

where $Q := \begin{bmatrix} 0 & M \\ M^{-1} & 0 \end{bmatrix}$.

We will say that the various interconnections are M-reversible when the basic building block is. We will also restrict ourselves to the case where R and U are unitary; the general case is treated in [29].

In a well-posed, M-reversible periodic interconnection, with $N = 2L$ units, signals flowing to the right of the L^{th} block are related to these flowing to the left from the $(L+1)^{th}$ block, provided the input d and the initial state $x_0 := x(t=0)$ are well-chosen. More precisely, one can show

PROPOSITION 4.1. *Let the input d and initial state x_0 of a well-posed M-reversible periodic system satisfy*

$$d(2L+1-s) = Rd(s)\ ;\ x_0(2L+1-s) = Px_0(s) \text{ for all } 1 \le s \le 2L\ .$$

Then

$$x(s) = Px(t, 2L+1-s) \tag{4.1}$$

$$v(s) = Qv(2L+1-s) \tag{4.2}$$

$$z(s) = Uz(2L+1-s) \text{ for all } 1 \le s \le 2L\ . \tag{4.3}$$

In particular $v_+(s=1) = Mv_-(s=2L) = Mw_-(s=1)$ and $v_-(s=L) = M^{-1}v_+(s=L+1) = M^{-1}w^+(s=L)$.

The following result also holds:

PROPOSITION 4.2. *Assume the basic building block is M-reversible. The finite extent interconnection with boundary conditions matrix M and L units is well-posed if the periodic interconnection with $N = 2L$ units is well-posed.*

A proof can be found in [29]. Combining Proposition 4.1 and 4.2 we obtain the following result:

THEOREM 4.2. *Let an integer L and an M-reversible building block be given. Assume that the periodic interconnection with $2L$ units is well-posed. Then*

1. *this periodic interconnection is stable if and only if the finite interconnection with boundary conditions matrix M and L units is stable.*
2. *The latter is contractive if the periodic interconnection is contractive.*

Theorems 4.1 and 4.2 provide a link between the properties of infinite and finite extent interconnections. In particular, they imply that the analysis conditions of Section 3 are also sufficient for the finite interconnection with boundary conditions problem, provided that the basic building block is M-reversible.

These results can readily be extended to more than one spatial dimension; the details may be found in [29].

5. Controller synthesis and implementation. For control design, the basic building block is augmented to include sensor and actuator variables, as depicted in Figure 11 for one spatial dimension. The governing equations, again for one spatial dimension, become:

$$\begin{bmatrix} \dot{x}^{G}(t,s) \\ w^{G}(t,s) \\ \begin{bmatrix} z(t,s) \\ y(t,s) \end{bmatrix} \end{bmatrix} = \begin{bmatrix} A^{G}_{TT} & A^{G}_{TS} & B^{G}_{T} \\ A^{G}_{ST} & A^{G}_{SS} & B^{G}_{S} \\ C^{G}_{T} & C^{G}_{S} & D^{G} \end{bmatrix} \begin{bmatrix} x^{G}(t,s) \\ v^{G}(t,s) \\ \begin{bmatrix} d(t,s) \\ u(t,s) \end{bmatrix} \end{bmatrix}. \tag{5.1}$$

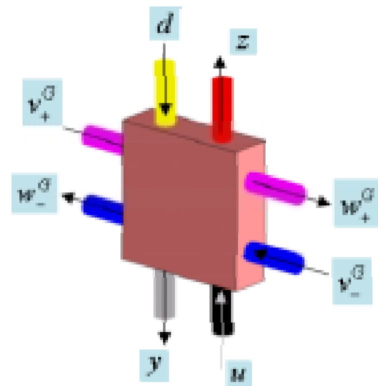

FIG. 11. *Basic building block for control design, one spatial dimension.*

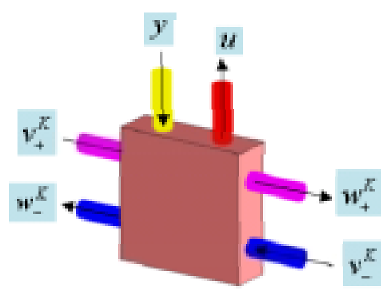

FIG. 12. *Basic building block for controller, one spatial dimension.*

The control design objective is to design a controller, depicted in Figure 12, with the following governing equations:

$$\begin{bmatrix} \dot{x}^{\mathrm{K}}(t,s) \\ w^{\mathrm{K}}(t,s) \\ u(t,s) \end{bmatrix} = \begin{bmatrix} A^{\mathrm{K}}_{\mathbf{TT}} & A^{\mathrm{K}}_{\mathbf{TS}} & B^{\mathrm{K}}_{\mathbf{T}} \\ A^{\mathrm{K}}_{\mathbf{ST}} & A^{\mathrm{K}}_{\mathbf{SS}} & B^{\mathrm{K}}_{\mathbf{S}} \\ C^{\mathrm{K}}_{\mathbf{T}} & C^{\mathrm{K}}_{\mathbf{S}} & D^{\mathrm{K}} \end{bmatrix} \begin{bmatrix} x^{\mathrm{K}}(t,s) \\ v^{\mathrm{K}}(t,s) \\ y(t,s) \end{bmatrix} \tag{5.2}$$

such that the closed loop system is well-posed, stable, and contractive. The resulting closed loop systems for various types of interconnections are depicted in Figure 13 (in the interest of clarity, the closed loop system inputs and outputs have been omitted from the diagram for two spatial dimensions).

Given the governing equations (5.1) for the open loop plant and for the candidate controller (5.2), one may readily construct a realization for the closed loop system (2.3); the details may be found in [16]. We may then invoke the analysis LMI in Section 3 to determine if the closed loop system

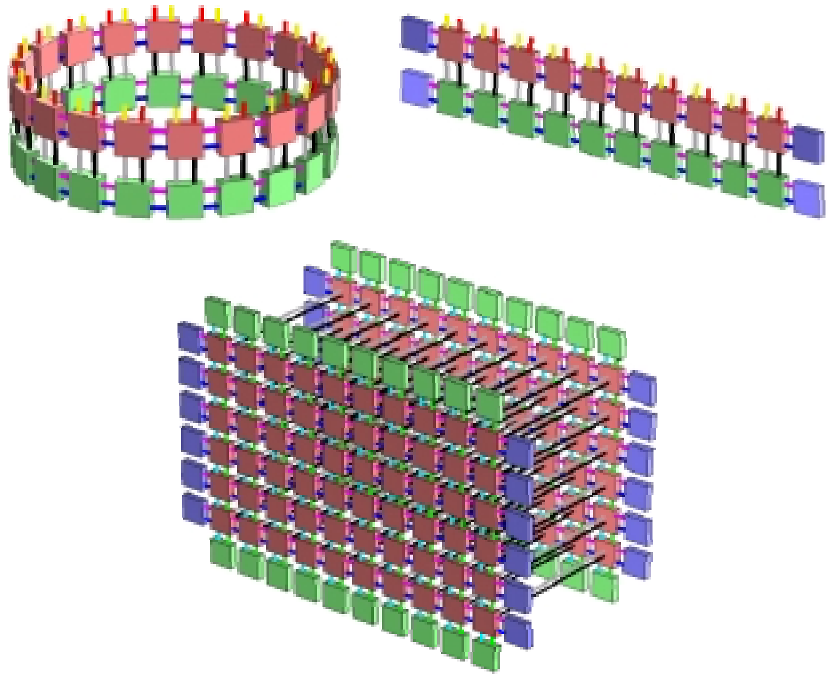

FIG. 13. *Closed loop systems.*

is well-posed, stable, and contractive. The complication is, of course, that the controller is not known a-priori, and must be designed.

Upon inspection of the condition in Theorem 3.1, it would seem that the control design problem is hopelessly non-convex, since the closed loop matrices are a function of the plant *and* the controller. As is shown in [9, 16], however, the synthesis problem can be made convex by performing several co-ordinate transformations. In particular, the synthesis problem can be expressed as an LMI *with no added conservatism*; the only conservatism incurred is from the analysis condition which is generally only sufficient, and not necessary. In other words, given the open loop plant equations (5.1), there exists a controller (5.2) such that the analysis LMI in Theorem 3.1 is satisfied for the closed loop system (2.3) if and only if there exists a solution to a synthesis LMI (whose details are omitted).

It is shown in [9, 16] that the size of the resulting controller is always less than or equal to that of the plant. In particular, the size of $x^K(t,s)$ is less than or equal to that of $x^G(t,s)$, etc.. It is shown in [29] that M-reversibility is also inherited by the controller.

Note that the controller implementation is distributed, as is depicted in Figure 13. Just like the plant, the controller consists of various identical subsystems, each of which is a linear, time invariant, finite dimensional state space system, which are interconnected to their nearest neighbors.

The practical advantages of such an implementation are obvious: the computation is distributed, and physical interconnections are localized. In addition, well-posedness, stability, and performance are guaranteed for *any* number of interconnected units, with obvious consequences for system re-configuration and fault-tolerance.

6. Example. Consider the following system equations, expressed in operator form for brevity:

$$\begin{aligned}
\ddot{p} &= \frac{1}{8}\left(\mathbf{S}_1+\mathbf{S}_1^{-1}+\mathbf{S}_2+\mathbf{S}_2^{-1}+4\right)p+\frac{1}{16}\left(\mathbf{S}_1+\mathbf{S}_1^{-1}-2\right)\left(\mathbf{S}_2+\mathbf{S}_2^{-1}-2\right)d_1+u \\
z_1 &= \frac{1}{16}\left(\mathbf{S}_1+\mathbf{S}_1^{-1}-2\right)\left(\mathbf{S}_2+\mathbf{S}_2^{-1}-2\right)p \\
z_2 &= u \\
y &= p + d_2 \ .
\end{aligned}$$

Each signal is a function of one temporal independent variable, and two spatial independent variables: $p = p(t, s_1, s_2)$. Recall that $(\mathbf{S}_1 p)(t, s_1, s_2) = p(t, s_1 + 1, s_2)$, $(\mathbf{S}_2 p)(t, s_1, s_2) = p(t, s_1, s_2 + 1)$, etc. The above equations can readily be expressed as per 5.1 using the software package described in [11]. The resulting realization has two temporal states $x(t, s_1, s_2)$ ($A_{\mathbf{TT}}$ is a two by two matrix), and each of the interconnection variables $v_{+,1}(t, s_1, s_2)$, $v_{-,1}(t, s_1, s_2)$, $v_{+,2}(t, s_1, s_2)$, and $v_{-,2}(t, s_1, s_2)$ is of size two ($A_{\mathbf{SS}}$ is an eight by eight matrix); the details are omitted.

Some things to note about the example:

1. The disturbance d_1 acts through a spatial high-pass filter. In particular, the filter completely rejects disturbances that are constant in space, but passes through disturbances whose entries alternate in sign with their nearest neighbors. For example, focusing in on a three by three grid, this high frequency disturbance would have the following profile:

$$d_1(t) = \begin{bmatrix} & & \vdots & & \\ & f(t) & -f(t) & f(t) & \\ \cdots & -f(t) & f(t) & -f(t) & \cdots \\ & f(t) & -f(t) & f(t) & \\ & & \vdots & & \end{bmatrix} \tag{6.1}$$

   where $f(t)$ is some function of time.
2. The same spatial filter is used to define error variable z_1. We are thus interested in rejecting high spatial frequency variations of variable p.

3. The second error variable z_2 is the control effort u. The sensor signal y is simply p corrupted by noise d_2. The control signal u acts directly on the $\ddot{p}$ equation.
4. The unforced dynamics

$$\ddot{p} = \frac{1}{8}\left(\mathbf{S}_1 + \mathbf{S}_1^{-1} + \mathbf{S}_2 + \mathbf{S}_2^{-1} + 4\right) p \tag{6.2}$$

   have a simple interpretation: the force on a mass particle at location (s_1, s_2), in a direction orthogonal to the grid, is a function of the difference between the displacements of the particle and its nearest neighbors, in a direction orthogonal to the grid, and is repulsive in nature.

This example is perhaps the simplest, non-trivial applications of the tools presented in this paper. In particular,

1. It is in two spatial dimensions. An explicit state-space representation of a ten by ten grid, for example, would result in a 200 by 200 state transition matrix.
2. Spatial filters are used to shape the input disturbance, and define the performance objective.

The price to be paid for this simplicity, however, is physical relevance. While one could readily ascribe a physical interpretation to the above equations (a lumped approximation of a membrane under compression, or electro-static forces acting on a two-dimensional array of charged particles), it would not be a realistic one. The reader is referred to [10, 23] for an application and a more realistic example tackled using these tools.

6.1. Distributed controller. A distributed controller was designed using the control synthesis software described in [11]. The resulting controller had one temporal state $x^{\mathrm{K}}(t, s_1, s_2)$, and each of the interconnection variables $v^{\mathrm{K}}_{+,1}(t, s_1, s_2)$, $v^{\mathrm{K}}_{-,1}(t, s_1, s_2)$, $v^{\mathrm{K}}_{+,2}(t, s_1, s_2)$, and $v^{\mathrm{K}}_{-,2}(t, s_1, s_2)$ was of size two.

It took 0.6 seconds to design the controller on a Pentium III, 1.13 GHz micro-processor. The upper bound to the $\mathcal{L}_2$ induced gain of the closed loop system, as provided by the controller synthesis routine, was 4.58. The $\mathcal{L}_2$ induced gain of the system was then calculated to be 4.20 using a frequency search (note that these figures do not have to match, since the analysis LMI in Section 3 is a sufficient, but not necessary, condition).

6.2. Decentralized controller number 1. A fully decentralized controller was then extracted from the distributed controller by discarding all interconnection variables. The resulting closed loop system was unstable.

6.3. Decentralized controller number 2. A fully decentralized controller was then designed by simplifying the system equations as follows:

$$\begin{aligned} \ddot{p} &= p - d_1 + u \\ z_1 &= -p \\ z_2 &= u \\ y &= p + d_2 \ . \end{aligned}$$

The simplification is obtained by considering the worst case effects of the spatial operators. In terms of the unforced dynamics, the most instability is obtained when all the neighbors are acting in unison. In terms of the disturbance and error variable, the worst case effects occur when neighbors alternate in sign.

The resulting controller was then interconnected with the open loop plant, and a frequency search used to determine the $\mathcal{L}_2$ gain. The result was 5.74.

6.4. Other decentralized control designs. Other decentralized controllers were designed by considering various simplifications of the system equations. They either resulted in an unstable closed loop system, or in a closed loop system with a larger $\mathcal{L}_2$ induced gain than that obtained with decentralized controller number 2.

6.5. Centralized controllers. Centralized controllers were designed for periodic interconnections of various size (corresponding to the torus in Figure 8) using the LMI toolbox [25]. The largest size problem that could be solved in a reasonable time was a 3 by 3 grid, which took 378 seconds. The resulting $\mathcal{L}_2$ induced gain was 4.02. The controller was a 9 state, 9 input and 9 output system.

The computation time for a 2 by 2 grid was 4.14 seconds, and for a 6 by 1 grid 44.95 seconds. By assuming a polynomial growth in computation time as a function of the size of the problem [3], it would take on the order of 5 years to design a centralized controller for a 10 by 10 grid (this does not take into account computer memory limitations).

6.6. Summary. For this particular example, the distributed controller resulted in a closed loop gain which was 1.37 times smaller than that obtained with the best decentralized controller, and 1.05 times larger than that obtained with a centralized controller for a three by three grid.

7. Concluding remarks. In this section we outline several directions in which this research is being expanded.

7.1. Nonlinear interconnected systems. Consider the case where the subsystem equations in 2.3 are replaced by *nonlinear* equations:

$$\begin{bmatrix} \dot{x}(t,s) \\ w(t,s) \\ z(t,s) \end{bmatrix} = \begin{bmatrix} f_{\mathbf{T}}(x(t,s), v(t,s), d(t,s)) \\ f_{\mathbf{S}}(x(t,s), v(t,s), d(t,s)) \\ h(x(t,s), v(t,s), d(t,s)) \end{bmatrix} . \tag{7.1}$$

As in Section 2, we can build various types of large-scale systems by interconnecting a number of such blocks. A significant amount of work has been devoted to the analysis and control of such non-linear spatially-interconnected systems. Most approaches have focused on *decentralized* control and often require that the coupling terms be small and/or bounded by some known function [40, 49, 45].

It might be possible to weaken these assumptions and/or achieve better performance properties in terms of disturbance rejection if one adopts a *distributed* control scheme similar to what have been presented here in the linear case. Other control problems for nonlinear interconnected systems that naturally lend themselves to a distributed architecture are feedback linearization and inversion-based trajectory planning.

This becomes apparent if one first considers the infinite extent interconnection in the case where the function $f_{\mathbf{S}}$ does not depend on v. After some easy manipulations such a system can be rewritten as

$$\dot{x}(t) = F(\mathbf{S}^k x(t), ..., \mathbf{S}^{-k} x(t), \mathbf{S}^l d(t), ..., \mathbf{S}^{-l} d(t)) \tag{7.2}$$

$$z(t) = H(\mathbf{S}^k x(t), ..., \mathbf{S}^{-k} x(t), \mathbf{S}^l d(t), ..., \mathbf{S}^{-l} d(t)) \tag{7.3}$$

where $\mathbf{S}$ is the spatial shift operator and k, l are positive integers.

System 7.2 is formally similar to a non-linear time-delay system, since *two* different operators –temporal differentiation and a spatial shift– appear; an important difference is that it is possible to shift backwards in space, but not in time.

It should thus be possible to generalize the methods developed for time-delay systems (like the concept of δ-freeness, [22], which generalizes flatness, [21] or the differential algebraic framework of [36, 33]) to handle inversion, path-planning and feedback linearization of nonlinear spatially interconnected systems. Such techniques would naturally yield a distributed control law since they involve taking successive shifts and differentiations of the state to determine the control law.

Current work focuses on incorporating the spatial structure of the plant into these design procedures, as it is in fact hidden in Equations 7.2.

7.2. Heterogeneous systems. Consider the case where the equations in 2.3 are replaced by the following *spatially varying* equations:

$$\begin{bmatrix} \dot{x}(t,s) \\ w(t,s) \\ z(t,s) \end{bmatrix} = \begin{bmatrix} A_{\mathbf{TT}}(s) & A_{\mathbf{TS}}(s) & B_{\mathbf{T}}(s) \\ A_{\mathbf{ST}}(s) & A_{\mathbf{SS}}(s) & B_{\mathbf{S}}(s) \\ C_{\mathbf{T}}(s) & C_{\mathbf{S}}(s) & D(s) \end{bmatrix} \begin{bmatrix} x(t,s) \\ v(t,s) \\ d(t,s) \end{bmatrix}. \tag{7.4}$$

In other words, the subsystems are no longer required to be identical. By incorporating the ideas and concepts introduced in this paper with the LMI synthesis techniques developed for linear time varying systems [20], LMI synthesis conditions for spatially varying systems can be obtained. These results are presented in [19, 17, 18].

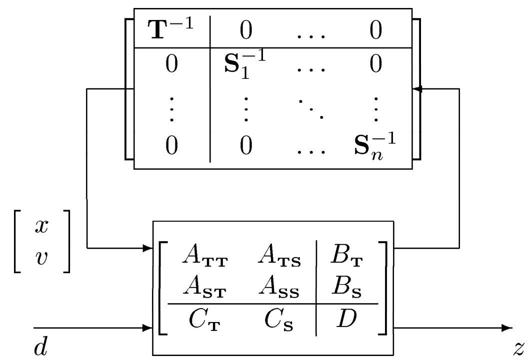

FIG. 14. *LFT representation of a multidimensional system.*

7.3. Connections to robust control. In this section, we mainly deal with the *analysis problem* for infinite extent systems. We also restrict ourselves to the special case in which the signal flow is in one direction only, that is, v_- and w_- have dimension zero. Denoting temporal differentiation by $\mathbf{T}$ and the spatial shift operators by $\mathbf{S}_i$, ($i = 1, 2, \ldots, n$ in the case of n spatial dimensions), we obtain

$$\left[\begin{array}{c|cccc} \mathbf{T} & 0 & \ldots & 0 \\ \hline 0 & \mathbf{S}_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbf{S}_n \end{array}\right] \left[\begin{array}{c} x \\ \hline v_1 \\ \vdots \\ v_n \end{array}\right] = \left[\begin{array}{c|c} A_{\mathbf{TT}} & A_{\mathbf{TS}} \\ \hline A_{\mathbf{ST}} & A_{\mathbf{SS}} \end{array}\right] \left[\begin{array}{c} x \\ \hline v_1 \\ \vdots \\ v_n \end{array}\right] + \left[\begin{array}{c} B_{\mathbf{T}} \\ \hline B_{\mathbf{S}} \end{array}\right] d,$$
$$z = \begin{bmatrix} C_{\mathbf{T}} & C_{\mathbf{S}} \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + Dd \tag{7.5}$$

where the arguments $(t, s_1, s_2, \ldots, s_n)$ of all the signals have been omitted for brevity, as have the multiplicities of operators $\mathbf{T}$ and $\mathbf{S}_i$.

It is seen at once that the above equations represent a linear fractional transformation (LFT) [50], [39] on the temporal and spatial shift operators. The block diagram of this LFT is shown in Figure 14. Since the LFT paradigm is a powerful and well-developed approach to problems involving LFTs on structured uncertainties [50], it is expected that the multidimensional control problem has many features in common with robust analysis against structured uncertainties. This is indeed the case, and we may make the following observations:

1. The necessary and sufficient conditions for robust stability of a linear time-invariant system against time-invariant contractive structured uncertainty require that the structured singular value μ of the system be bounded above by 1. The LFT formulation suggests that the multidimensional analysis problem is very similar to a μ

problem. This forms the basis for the work in [26]; see Section 1. In fact, it reduces exactly to a μ problem with scalar blocks if the spatial coordinates are assumed to be causal. The fact that the scaled small-gain conditions of this paper are not necessary for robust stability is therefore a consequence of the result that μ is not equal to its upper bound for two or more scalar block uncertainties [39] in this case. However, computation of μ is in general a very hard problem. In fact, certain cases of it have been proved to be NP-hard [4], [47]. It is therefore very unlikely that general, computationally tractable necessary and sufficient conditions for the multidimensional problem will emerge in the near future.

2. In both cases, we make a very similar relaxation by replacing the exact problem by a scaled small-gain condition. In the usual case where the LFT is over structured contractive uncertainties, we obtain positive definite scaling matrices. However, in the case of multidimensional systems, the spatial shift operators are restricted to be unitary, and we thus have greater freedom in choosing our scales, i.e., we can choose them to be arbitrary symmetric matrices.
3. In the case of structured, linear time varying uncertainties, arguments based on the S-procedure [34] show that the scaled small-gain condition is not only sufficient but also necessary for robust performance. It is a very interesting problem to determine whether a similar condition holds for multidimensional analysis. We shall comment on this problem in greater detail shortly.
4. The LFT approach suggests a very simple way of taking into account uncertainties in the state-space entries in addition to the spatial operators. By the standard method of "pulling out uncertainties" [32], we can extract an LFT structure that includes both spatial shifts and the contractive uncertainties. It is almost obvious that if we solve a scaled small-gain theorem for scalings that commute with the uncertainties and are positive definite, in addition to the scalings that commute with the shift operators, we are guaranteed robust performance. This yields an approach to handle structured perturbations in multidimensional systems.

Thus our approach to multidimensional analysis and synthesis relies on computationally attractive sufficient conditions which may be conservative. Therefore it is of interest to estimate qualitatively or quantitatively this conservatism. Consider the system shown in Figure 14. In the case where there is only one spatial dimension, arguments based on the S-procedure can be used to prove the following result:

PROPOSITION 7.1. *([14]) Given that the matrix $A_{\mathbf{TT}}$ is Hurwitz, the LMI condition in Theorem 3.1 is necessary and sufficient for robust performance of the multidimensional system if the spatial shift operator is replaced by an arbitrary unitary operator which is allowed to be spatially varying (i.e., which does not necessarily commute with the spatial shift operator).*

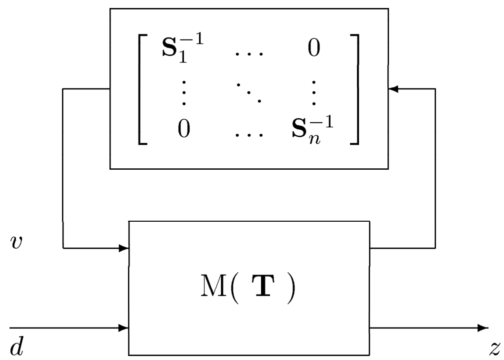

FIG. 15. *LFT representation to apply the S-procedure.*

This result is obtained by absorbing the temporal operator of the system into the block $M(\mathbf{T})$ (see Figure 15) and considering the multidimensional system as an LFT of the LTI system $M(\mathbf{T})$ and the spatial operators, and applying S-losslessness type arguments. We require that the LTI part of the system be stable for this analysis, and we therefore need to investigate under what conditions stability of the multidimensional system implies that of the LTI component. In the case where the number of spatial dimensions is greater than one, we conjecture that, given the stability of the LTI part of the system, the LMI conditions in Theorem 3.1 are necessary and sufficient if the spatial shift operators $\mathbf{S}_i$ are replaced by arbitrary unitary operators δ_i that commute with all $\mathbf{S}_j$ for $j \neq i$ (but do not necessarily commute with $\mathbf{S}_i$). Physically, this means that it is sufficient to consider operators each of which can transfer power in one spatial dimension only.

REFERENCES

[1] B. BAMIEH, F. PAGANINI, AND M. DAHLEH. Distributed control of spatially invariant systems. *IEEE Transactions on Automatic Control*, **47**(7): 1091–1118, 2002.

[2] S.P. BANKS. *State-space and frequency domain methods in the control of distributed parameter systems.* Peter Peregrinus, 1983.

[3] S.P. BOYD, L. EL GHAOUI, E. FERON, AND V BALAKRISHNAN. *Linear Matrix Inequalities in System and Control Theory.* SIAM studies in applied mathematics. Philadelphia: Society for Industrial and Applied Mathematics, 1994.

[4] R. BRAATZ, J. DOYLE, P. YOUNG, AND M. MORARI. Computational complexity of μ calculation. *IEEE Transactions on Automatic Control*, **39**: 1000–1002, 1994.

[5] R.W. BROCKETT AND J.L. WILLEMS. Discretized partial differential equations: Examples of control systems defined on modules. *Automatica*, **10**: 507–515, 1974.

[6] R.S. CHANDRA, J. FOWLER, AND R. D'ANDREA. Control of interconnected systems of finite spatial extent. *IEEE Conference on Decision and Control*, 2002. To appear.

[7] D.F. Chichka and J.L. Speyer. Solar-powered, formation-enhanced aerial vehicle system for sustained endurance. In *Proc. American Control Conference*, pp. 684–688, 1998.

[8] R.F. Curtain and H.J. Zwart. *An Introduction to Infinite-Dimensional Linear Systems Theory.* Springer-Verlag, 1995.

[9] R. D'Andrea. A linear matrix inequality approach to decentralized control of distributed parameter systems. In *Proc. American Control Conference*, pp. 1350–1354, 1998.

[10] R. D'Andrea. Linear matrix inequalities, multidimensional system optimization, and control of spatially distributed systems: An example. In *Proc. American Control Conference*, pp. 2713–2717, 1999.

[11] R. D'Andrea. Software for modeling, analysis, and control design for multidimensional systems. In *IEEE International Symposium on Computer-Aided Control System Design*, pp. 24–27, 1999.

[12] R. D'Andrea. Extension of Parrott's theorem to non-definite scalings. *IEEE Transactions on Automatic Control*, **45**(5): 937–940, 2000.

[13] R. D'Andrea and C. Beck. Temporal discretization of spatially distributed systems. In *Proc. IEEE Conference on Decision and Control*, pp. 197–202, 1999.

[14] R. D'andrea and R.S. Chandra. Multidimensional robust analysis. *American Control Conference*, 2003. Submitted.

[15] R. D'Andrea, G. Dullerud, and S. Lall. Convex ℓ_2 synthesis for multidimensional systems. In *Proc. IEEE Conference on Decision and Control*, pp. 1883–1888, 1998.

[16] R. D'Andrea and G.E. Dullerud. Distributed control of spatially interconnected systems. *IEEE Transactions on Automatic Control.* Submitted for publication.

[17] G. Dullerud and R. D'Andrea. Distributed control of inhomogeneous systems, with boundary conditions. In *Proc. IEEE Conference on Decision and Control*, pp. 186–190, 1999.

[18] G.E. Dullerud and R. D'Andrea. Distributed control of heterogeneous systems. *IEEE Transactions on Automatic Control.* Submitted for Publication.

[19] G.E. Dullerud, R. D'Andrea, and S. Lall. Control of spatially varying distributed systems. In *Proc. IEEE Conference on Decision and Control*, pp. 1889–1893, 1998.

[20] G.E. Dullerud and S. Lall. A new approach for analysis and synthesis of time-varying systems. *IEEE Transactions on Automatic Control*, **44**(8): 1486–1497, 1999.

[21] M. Fliess, J. Lévine, P. Martin, and P. Rouchon. Flatness and defect of nonlinear systems: introductory theory and examples. *Internat. J. Control*, **61**(6): 1327–1361, 1995.

[22] M. Fliess and H. Mounier. Quelques propriétés structurelles des systèmes linéaires à retards constants. *C.R. Acad. Sci. Paris*, **319**: 289–294, 1994.

[23] J.M. Fowler and R. D'Andrea. Distributed control of close formation flight. In *Proc. IEEE Conference on Decision and Control*, 2002.

[24] P. Gahinet and P. Apkarian. A linear matrix inequality approach to H-infinity control. *International Journal of Robust and Nonlinear Control*, **4**: 421–448, 1994.

[25] P. Gahinet, A. Nemirovskii, A. Laub, and M. Chilali. *The LMI Control Toolbox.* The MathWorks Inc., 1995.

[26] D. Gorinevsky and G. Stein. Structured uncertainty analysis of robust stability for spatially distributed systems. *IEEE Conference on Decision and Control*, 2000.

[27] T. Iwasaki and R.E. Skelton. All controllers for the general h-infinity control problem: LMI existence conditions and state space formulas. *Automatica*, **30**(8), 1994.

[28] P.P. Khargonekar, K. Poolla, and A. Tannebaum. Robust control of linear time-invariant plants using periodic compensation. *IEEE Transactions on Automatic Control*, **27**(3): 627–638, 1982.

[29] C. Langbort and R. D'Andrea. Distributed control of spatially reversible interconnected systems with boundary conditions. *SIAM Journal of Control and Optimization*. Submitted for publication.

[30] C. Langbort and R. D'Andrea. Imposing boundary conditions for spatially interconnected systems. In *Proc. American Control Conference*, 2002. Submitted for publication.

[31] D. Laughlin, M. Morari, and R.D. Braatz. Robust performance of cross-directional basis-weight control in paper machines. *Automatica*, **29**(6): 1395–1410, 1993.

[32] J.M. Maciejowski. *Multivariable feedback design*. Addison-Wesley, 1989.

[33] L.A. Márquez-Martinez, C. H. Moog, and M. Velasco-Villa. The structure of nonlinear time delay systems. In Proc. of the 6th IEEE Mediterranean Conference on Control and Systems, MCCS'98 Sardinia, 1998.

[34] A. Megretski and A. Rantzer. System analysis via integral quadratic constraints. *IEEE Transactions on Automatic Control*, **42**(6): 819–830, 1997.

[35] S.M. Melzer and B.C. Kuo. Optimal regulation of systems described by a countably infinite number of objects. *Automatica*, **7**: 359–366, 1971.

[36] C.H. Moog, R. Castro-Linares, M. Velasco-Villa, and L.A. Marquez-Martinez. The disturbance decoupling problem for time-delay nonlinear systems. *IEEE Transactions on Automatic Control*, **45**(2): 305–309, 2000.

[37] M. Napoli and B. Bamieh. Modeling and observed design for an array of electrostatically actuated microcantilevers. In *Proc. IEEE Conference on Decision and Control*, pp. 4274–4279, 2001.

[38] A. Packard. Gain scheduling via linear fractional transformations. *Systems and Control Letters*, **22**(2): 79–92, 1994.

[39] A. Packard and J.C. Doyle. The complex structured singular value. *Automatica*, **29**(1): 71–109, 1993.

[40] A. Pant, P. Seiler, and J.K. Hedrick. Mesh stability of look-ahead interconnected systems. *IEEE Transactions on Automatic Control*, **47**(2): 403–407, 2002.

[41] A. Rantzer. On the Kalman-Yakubovich-Popov lemma. *Systems and Control Letters*, **28**(1): 7–10, 1996.

[42] H. Raza and P. Ioannou. Vehicle following control design for automated highway systems. *IEEE Control Systems*, **16**(6): 43–60, 1996.

[43] G.B. Shaw, D.W. Miller, and D.E. Hastings. *The Generalized Information Network Analysis Methodology for Distributed Satellite Systems*. PhD thesis, Massachusetts Institute of Technology, 1998.

[44] G.E. Stewart. *Two dimensional loop shaping controller design for paper machine cross-directional processes*. PhD thesis, University of British Columbia, 2000.

[45] D. Swaroop and J.K. Hedrick. String stability for a class of nonlinear systems. *IEEE Transactions on Automatic Control*, **41**: 349–357, 1996.

[46] M.E. Taylor. *Partial Differential Equations I: Basic Theory*. Springer Verlag, 1996.

[47] O. Toker and H. Ozbay. On the complexity of purely complex mu computation and related problems in multidimensional systems. *IEEE Transactions on Automatic Control*, **43**(3): 409–414, 1998.

[48] J.D. Wolfe, D.F. Chichka, and J.L. Speyer. Decentralized controllers for unmmaned aerial vehicle formation flight. *American Institute of Aeronautics and Astronautics*, 96-3833, 1996.

[49] X.G. Yan and G.Z. Dai. Decentralized output feedback robust control for nonlinear large-scale systems. *Automatica*, **34**(11): 1469–1472, 1998.

[50] K. Zhou, J.C. Doyle, and K. Glover. *Robust and Optimal Control*. Prentice Hall, 1995.