Distributed Control Design for Spatially Interconnected Systems

Raffaello D'Andrea and Geir E. Dullerud

***Abstract*—This paper deals with analysis, synthesis, and implementation of distributed controllers, designed for spatially interconnected systems. We develop a state space framework for posing problems of this type, and focus on systems whose model is spatially discrete. In this paper, analysis and synthesis results are developed for this class of systems using the l_2-induced norm as the performance criterion. The results are stated in terms of linear matrix inequalities and are thus readily amenable to computation. A special implementation of the resulting controllers is presented, which is particularly attractive for distributed operation of the controller. Several examples are provided to further illustrate the application of the results.**

***Index Terms*—Distributed control, H_∞, interconnected systems, linear matrix inequalities (LMIs).**

I. Introduction

MANY systems consist of similar units which directly interact with their nearest neighbors. Even when these units have tractable models and interact with their neighbors in a simple and predictable fashion, the resulting system often displays rich and complex behavior when viewed as a whole. There are many examples of such systems, including automated highway systems [37], airplane formation flight [42], [9], satellite constellations [39], cross-directional control in paper processing applications [40], and very recently, micro-cantilever array control for massively parallel data storage [31]. One can also consider lumped approximations of partial differential equations (PDEs)—examples include the deflection of beams, plates, and membranes, and the temperature distribution of thermally conductive materials [41].

An important aspect of many of these systems is that sensing and actuation capabilities exist at every unit. In the examples above, this is clearly the case for automated highway systems, airplane formation flight, satellite constellations, and cross-directional control systems. With the rapid advances in micro electromechanical actuators and sensors, however, we will soon be able to instrument systems governed by partial differential equations with distributed arrays of actuators and sensors, rendering lumped approximations with collocated sensors and actuators valid mathematical abstractions.

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If one attempts to control these systems using standard control design techniques, severe limitations will quickly be encountered as most optimal control techniques cannot handle systems of very high dimension *and* with a large number of inputs and outputs. It is also not feasible to control these systems with centralized schemes—the typical outcome of most optimal control design techniques—as these require high levels of connectivity, impose a substantial computational burden, and are typically more sensitive to failures and modeling errors than decentralized schemes.

In order for any optimal control technique to be successful, the structure of the system must be exploited in order to obtain tractable algorithms. In this paper, we present a state-space approach to controlling systems with a highly structured interconnection topology; in particular, we consider linear, spatially invariant systems that can be captured as fractional transformations on temporal and spatial operators. By doing so, many standard results in control—such as the Kalman–Yakubovich–Popov (KYP) Lemma, H_∞ optimization, and robustness analysis—can be generalized accordingly. The state space formulation yields conditions that can be expressed as linear matrix inequalities (LMIs) [7], resulting in tractable computational tools for control design and analysis.

The types of problems considered in this paper have a long history. In [30], optimal regulation for a countably infinite number of objects is considered by employing a bilateral Z-transform, which is analogous to the spatial shift operators introduced in this paper. In [8], it was shown that discretization of certain classes of PDEs result in control systems defined on modules, and that the resulting structure can be exploited to reduce computational effort.

Recently, [3], control problems for spatially invariant systems with quadratic performance criteria (such as $\mathcal{H}_2$ and $\mathcal{H}_\infty$) are tackled by extending familiar frequency-domain concepts for one-dimensional systems. The control design problem is then solved for a parameterized (over frequency) system of finite-dimensional systems. It is also shown that the optimal controller has a degree of spatial localization (similar to the plant) and can therefore be implemented in a distributed fashion.

Robust stability analysis problems for multidimensional systems are considered in [26]. Results are derived using Laplace transforms in several complex variables which show that the problem can be solved by the methods of structured uncertainty analysis (μ analysis) [33].

Cross-directional control of paper machine processes is considered in [40]. The notion of loop shaping [29] is extended to two-dimensional systems (one temporal, one spatial). The special structure of the paper machine problem, and of similar problems, is exploited to apply the results in [3] and obtain a computationally attractive practical control design methodology to address performance and robustness issues.

The paper is organized as follows. We introduce the notation and the basic concepts used throughout the paper in Section II. The class of systems considered in this paper are described in Section III. Analysis conditions are then presented in Section IV, which are used in Section V for controller design. An in-depth discussion on controller implementation may be found in Section VI. Several examples are included in Section VII, and concluding remarks are found in Section VIII.

II. Preliminaries

The set of integers is denoted $\mathbb{Z}$. The set of real numbers is denoted $\mathbb{R}$; $\mathbb{R}^+$ denotes the nonnegative subset. The notation $v \in \mathbb{R}^\bullet$ will be used to denote real valued, finite vectors whose size is either clear from context or not relevant to the discussion. We will often use the short-hand notation $v = (v_1, v_2)$ to capture a vector with several (not necessarily scalar) components.

The space of n by m matrices is denoted $\mathbb{R}^{n\times m}$; the space of symmetric n by n matrices is denoted $\mathbb{R}_{\mathbf{S}}^{n\times n}$. The n by n identity matrix is denoted I_n; when the dimension is clear from context, it is simply denoted I. Given real symmetric matrix $M, M > 0 (\geq 0)$ denotes property $v^*Mv > 0 (\geq 0)$ for all $v \neq 0$. The maximum singular value of a matrix A is denoted $\bar{\sigma}(A)$.

We are dealing with signals which are vector valued functions indexed by $L+1$ independent variables: $d = d(t, s_1, \ldots, s_L)$, where t denotes the temporal variable, and the s_i the spatial variables. We restrict ourselves to continuous time systems, and take t to be in $\mathbb{R}^+$. We also restrict ourselves to lumped systems, and only consider s_i that are integer valued. In particular, each of the s_i can either be in $\mathbb{Z}$, which captures infinite spatial extent in dimension i, or in some finite set $\{1, \ldots, N_i\}$, which captures periodicity of period N_i in spatial dimension i. We thus take s_i to be in $\mathbb{D}_i$, where $\mathbb{D}_i$ is either $\mathbb{Z}$ or $\{1, \ldots, N_i\}$. When $s_i \in \{1, \ldots, N_i\}$, modular arithmetic is used: $N_i + 1 := 1$. The L-tuple $(s_1, \ldots, s_L)$ is denoted by $\mathbf{s}$.

We often consider signals at a fixed time; it is thus convenient to separate the spatial and temporal parts of a signal, which motivates the following definitions:

Definition 1: The space ℓ_2 is the set of functions mapping $\mathbb{D}_1 \times \cdots \times \mathbb{D}_L$ to $\mathbb{R}^\bullet$ for which the following quantity is finite:

$$\sum_{s_1\in\mathbb{D}_1} \cdots \sum_{s_L\in\mathbb{D}_L} x(\mathbf{s})^* x(\mathbf{s}). \tag{1}$$

The inner product on ℓ_2 is defined as

$$\langle x, y\rangle_{\ell_2} := \sum_{s_1\in\mathbb{D}_1} \cdots \sum_{s_L\in\mathbb{D}_L} x(\mathbf{s})^* y(\mathbf{s}) \tag{2}$$

with corresponding norm $\|x\|_{\ell_2} := \sqrt{\langle x, x\rangle_{\ell_2}}$.

Definition 2: The space $\mathcal{L}_2$ is the set of functions mapping $\mathbb{R}^+$ to ℓ_2 for which the following quantity is finite:

$$\int_0^\infty \|u(t)\|_{\ell_2}^2 \, dt. \tag{3}$$

The inner product on $\mathcal{L}_2$ is defined as

$$\langle u, v\rangle_{\mathcal{L}_2} := \int_0^\infty \langle u(t), v(t)\rangle_{\ell_2} \, dt \tag{4}$$

with corresponding norm $\|u\|_{\mathcal{L}_2} := \sqrt{\langle u, u\rangle_{\mathcal{L}_2}}$.

With a slight abuse of notation, a signal $u \in \mathcal{L}_2$ can thus be considered a function of $L+1$ independent variables, $u = u(t, s_1, \ldots, s_L)$. Thus, for fixed t and $\mathbf{s}$, $u(t)$ is an element of ℓ_2 and $u(t, \mathbf{s})$ is a real-valued vector.

Let $u(t, \mathbf{s})$ be scalar valued. We can define the spatial shift operators $\mathbf{S}_i$, acting on signals in ℓ_2, as follows:

$$(\mathbf{S}_i u(t))(\mathbf{s}) := u(t, s_1, \ldots, s_i + 1, \ldots, s_L), \qquad i = 1, \ldots, L. \tag{5}$$

While we always work with signals that have a finite spatial norm at any instant in time, we sometimes work with signals that do not have a finite overall norm. We thus define $\mathcal{L}$ to be the set of functions mapping $\mathbb{R}^+$ to ℓ_2 for which the following quantity is finite for every $T \geq 0$:

$$\int_0^T \|u(t)\|_{\ell_2}^2 \, dt. \tag{6}$$

An operator $\mathbf{F}$ on ℓ_2 is said to be bounded if

$$\|\mathbf{F}\|_{\ell_2} := \sup_{x\in\ell_2, x\neq 0} \frac{\|\mathbf{F}x\|_{\ell_2}}{\|x\|_{\ell_2}} < \infty \tag{7}$$

where $\|\mathbf{F}\|_{\ell_2}$ is used to denote the induced gain of operator $\mathbf{F}$. The adjoint of a bounded operator $\mathbf{F}$ is denoted $\mathbf{F}^*$, and is the unique operator which satisfies $\langle u, \mathbf{F}v\rangle_{\ell_2} = \langle \mathbf{F}^* u, v\rangle_{\ell_2}$ for all $u, v \in \ell_2$. A bounded operator $\mathbf{F}$ is said to be invertible on ℓ_2 if there exist bounded operators $\mathbf{H}_{\mathrm{L}}$ and $\mathbf{H}_{\mathrm{R}}$ such that $\mathbf{H}_{\mathrm{L}}\mathbf{F}$ and $\mathbf{F}\mathbf{H}_{\mathrm{R}}$ are the identity operators. Similar definitions hold for operators on $\mathcal{L}_2$.

It is useful to extend the definition of an operator $\mathbf{F}$ on ℓ_2 to the space $\mathcal{L}$ in the following natural way:

$$(\mathbf{F}u)(t) := \mathbf{F}u(t) \qquad \forall u \in \mathcal{L} \qquad \forall t \in \mathbb{R}. \tag{8}$$

For example, $(\mathbf{S}_i u)(t, \mathbf{s}) := u(t, s_1, \ldots, s_i + 1, \ldots, s_L)$.

III. Interconnected Systems

We next introduce the systems considered in this paper. In the interest of clarity, we first present the relevant definitions and results for systems in one spatial dimension, i.e., the signals in question are of the form $d = d(t, \mathbf{s})$, where $\mathbf{s} \in \mathbb{D}$. Extensions to more than one spatial dimension are deferred to the end of this section.

A. Periodic and Infinite Interconnections

Consider the diagram in Fig. 1. It consists of a finite dimensional, linear time-invariant system governed by the following state-space equations:

$$\begin{bmatrix} \dot{x}(t,\mathbf{s}) \\ w(t,\mathbf{s}) \\ z(t,\mathbf{s}) \end{bmatrix} = \begin{bmatrix} A_{\mathbf{TT}} & A_{\mathbf{TS}} & B_{\mathbf{T}} \\ A_{\mathbf{ST}} & A_{\mathbf{SS}} & B_{\mathbf{S}} \\ C_{\mathbf{T}} & C_{\mathbf{S}} & D \end{bmatrix} \begin{bmatrix} x(t,\mathbf{s}) \\ v(t,\mathbf{s}) \\ d(t,\mathbf{s}) \end{bmatrix}$$
$$x(0,\mathbf{s}) = x_0(\mathbf{s}) \in \mathbb{R}^\bullet \tag{9}$$

where

$$v(t,\mathbf{s}) = \begin{bmatrix} v_+(t,\mathbf{s}) \\ v_-(t,\mathbf{s}) \end{bmatrix} \quad w(t,\mathbf{s}) = \begin{bmatrix} w_+(t,\mathbf{s}) \\ w_-(t,\mathbf{s}) \end{bmatrix} \tag{10}$$

and $\mathbf{s}$ is fixed. We assume that $v_+(t,\mathbf{s})$ and $w_+(t,\mathbf{s})$ are the same size, and that $v_-(t,\mathbf{s})$ and $w_-(t,\mathbf{s})$ are the same size. We next consider two types of interconnections based on identical copies of the basic building depicted in Fig. 1.

1) Periodic Interconnection: Let the number of units be N : $1 \le \mathbf{s} \le N$. Define a *periodic interconnection* as follows:

$$v_+(\mathbf{s}+1) = w_+(\mathbf{s}), \qquad 1 \le \mathbf{s} \le N-1 \tag{11}$$
$$v_+(\mathbf{s}=1) = w_+(\mathbf{s}=N) \tag{12}$$
$$v_-(\mathbf{s}-1) = w_-(\mathbf{s}), \qquad 2 \le \mathbf{s} \le N \tag{13}$$
$$v_-(\mathbf{s}=N) = w_-(\mathbf{s}=1). \tag{14}$$

This interconnection is depicted for $\mathrm{N} = 20$ in Fig. 2. Once the interconnection has been formed, the system inputs are simply d, and the system outputs are z; v and w can be considered internal system variables.

2) Infinite Interconnection: Consider an infinite number of units, interconnected as follows:

$$v_+(\mathbf{s}+1) = w_+(\mathbf{s}) \quad v_-(\mathbf{s}-1) = w_-(\mathbf{s}) \qquad \forall \mathbf{s} \in \mathbb{Z} \tag{15}$$

This is depicted in Fig. 3. This type of interconnection is similar to the one considered in [30], where a control system is designed for an infinite number of vehicles. As was pointed out in [30], and more recently in [3], an infinite approximation may be sufficient when dealing with a large number of systems. In particular, the scale of influence of localized effects is often much less than the scale of the whole system. Even if the uncontrolled system does not satisfy this property, it is likely that the controlled system will.

There is another important reason for considering infinite extent system; as is discussed in Section IV-D, if the infinite extent system is well-posed, stable, and contractive—notions to be defined in Section IV—these properties are inherited by *all* periodic interconnections, irrespective of the number of subsystems.

B. System Realization

In (9) and (10), let m_0 denote the size of the subsystem states $x(t,\mathbf{s}), m_+$ the size of interconnection variables $v_+(t,\mathbf{s})$ and $w_+(t,\mathbf{s})$, and m_- the size of interconnection variables $v_-(t,\mathbf{s})$ and $w_-(t,\mathbf{s})$. Let $\mathbf{m} = (m_0, m_+, m_-)$, and define the following structured operator on ℓ_2:

$$\mathbf{\Delta}_{\mathbf{S},\mathbf{m}} := \begin{bmatrix} \mathbf{S}I_{m_+} & 0 \\ 0 & \mathbf{S}^{-1}I_{m_-} \end{bmatrix}. \tag{16}$$

The role of m_0 in $\mathbf{m}$ will become apparent shortly. Note that we can now express the interconnection as $(\mathbf{\Delta}_{\mathbf{S},\mathbf{m}} v(t))(\mathbf{s}) = w(t,\mathbf{s})$, and we may thus write the interconnected system as follows:

$$\begin{bmatrix} \dot{x}(t,\mathbf{s}) \\ (\mathbf{\Delta}_{\mathbf{S},\mathbf{m}} v(t))(\mathbf{s}) \\ z(t,\mathbf{s}) \end{bmatrix} = \begin{bmatrix} A_{\mathbf{TT}} & A_{\mathbf{TS}} & B_{\mathbf{T}} \\ A_{\mathbf{ST}} & A_{\mathbf{SS}} & B_{\mathbf{S}} \\ C_{\mathbf{T}} & C_{\mathbf{S}} & D \end{bmatrix} \begin{bmatrix} x(t,\mathbf{s}) \\ v(t,\mathbf{s}) \\ d(t,\mathbf{s}) \end{bmatrix}$$
$$x(0,\mathbf{s}) = x_0(\mathbf{s}). \tag{17}$$

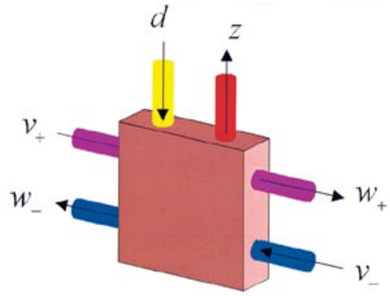

Fig. 1. Basic building block, one spatial dimension.

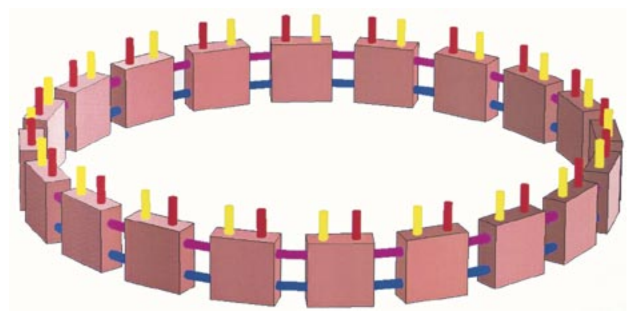

Fig. 2. Periodic interconnection.

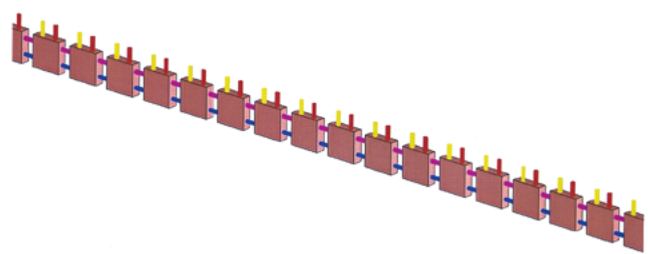

Fig. 3. Infinite interconnection.

Unless explicitly stated, we will assume that the initial condition for the state $x(0,\mathbf{s}) = 0$. By eliminating interconnection variables v, we can express the system as

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}d(t) \tag{18}$$
$$z(t) = \mathbf{C}x(t) + \mathbf{D}d(t) \tag{19}$$

where

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} := \begin{bmatrix} A_{\mathbf{TT}} & B_{\mathbf{T}} \\ C_{\mathbf{T}} & D \end{bmatrix} + \begin{bmatrix} A_{\mathbf{TS}} \\ C_{\mathbf{S}} \end{bmatrix} \times (\mathbf{\Delta}_{\mathbf{S},\mathbf{m}} - A_{\mathbf{SS}})^{-1} [A_{\mathbf{ST}} \quad B_{\mathbf{S}}] \tag{20}$$

and it is assumed that $(\mathbf{\Delta}_{\mathbf{S},\mathbf{m}} - A_{\mathbf{SS}})$ is invertible on ℓ_2. This assumption is equivalent to assuming that the interconnection is *well-posed*, a physically motivated concept that is formalized in Section IV-A. Note that well-posedness implies that operators $\mathbf{A}, \mathbf{B}, \mathbf{C}$, and $\mathbf{D}$ are bounded. Define

$$A := \begin{bmatrix} A_{\mathbf{TT}} & A_{\mathbf{TS}} \\ A_{\mathbf{ST}} & A_{\mathbf{SS}} \end{bmatrix} \quad B := \begin{bmatrix} B_{\mathbf{T}} \\ B_{\mathbf{S}} \end{bmatrix} \quad C := [C_{\mathbf{T}} \quad C_{\mathbf{S}}] \tag{21}$$

and

$$\mathbf{\Delta}_{\mathbf{m}} := \mathbf{diag}\left(\frac{d}{dt} I_{m_0}, \mathbf{S}I_{m_+}, \mathbf{S}^{-1}I_{m_-}\right) \tag{22}$$

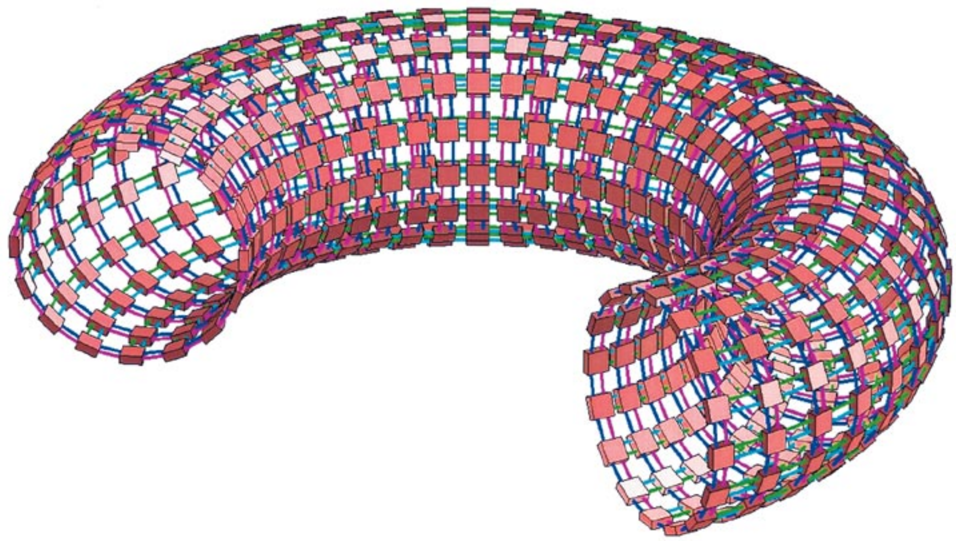

Fig. 4. Periodic interconnection in both spatial dimensions.

where operator $\mathbf{S}$ is extended to $\mathcal{L}$, as per (8). Given constant matrices A, B, C, and D of compatible dimension, and three-tuple $\mathbf{m}$, we may thus express a system in the following succinct form:

$$\begin{bmatrix} (\mathbf{\Delta_m} r)(t,\mathbf{s}) \\ z(t,\mathbf{s}) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} r(t,\mathbf{s}) \\ d(t,\mathbf{s}) \end{bmatrix} \tag{23}$$

where $r(t,\mathbf{s}) = (x(t,\mathbf{s}), v(t,\mathbf{s}))$. The system generated by A, B, C, D, and $\mathbf{m}$ is denoted $\mathcal{M} := \{A, B, C, D, \mathbf{m}\}$. In order to simplify notation, we will use $\mathcal{M}$ to denote both the system of equations generated by A, B, C, D, and $\mathbf{m}$, *and* the actual matrices A, B, C, D and tuple $\mathbf{m}$. It will be clear from context which meaning is being ascribed to $\mathcal{M}$.

C. Interconnected Systems in Higher Dimensions

The basic building block depicted in Fig. 1, and the periodic and infinite interconnections described earlier, can readily be extended to more than one spatial dimension. For example, in two dimensions $\mathbf{s} = (s_1, s_2)$ and

$$v(t,s_1,s_2) = \left(\begin{bmatrix} v_{+,1}(t,s_1,s_2) \\ v_{-,1}(t,s_1,s_2) \end{bmatrix}, \begin{bmatrix} v_{+,2}(t,s_1,s_2) \\ v_{-,2}(t,s_1,s_2) \end{bmatrix} \right) \tag{24}$$

$$w(t,s_1,s_2) = \left(\begin{bmatrix} w_{+,1}(t,s_1,s_2) \\ w_{-,1}(t,s_1,s_2) \end{bmatrix}, \begin{bmatrix} w_{+,2}(t,s_1,s_2) \\ w_{-,2}(t,s_1,s_2) \end{bmatrix} \right). \tag{25}$$

Various interconnections can then be defined; the details are omitted. For example, a periodic interconnection in both spatial dimensions is depicted in Fig. 4—in the interest of clarity, only a portion of the resulting torus is depicted and the inputs d and the outputs z have been omitted from the diagram.

In terms of the realization of a system, we can proceed as follows. For a given $\mathbf{m}$

$$(m_0, m_1, m_{-1}, m_2, m_{-2}, \ldots, m_L, m_{-L}) \tag{26}$$

$\mathbf{\Delta_{S,m}}$ is defined as

$$\mathbf{\Delta_{S,m}} := \mathbf{diag}\left(\mathbf{S}_1 I_{m_1}, \mathbf{S}_1^{-1} I_{m_{-1}}, \mathbf{S}_2 I_{m_2}, \mathbf{S}_2^{-1} I_{m_{-2}} \ldots, \mathbf{S}_L^{-1} I_{m_{-L}}\right). \tag{27}$$

Similarly, $\mathbf{\Delta_m}$ is defined as follows:

$$\mathbf{\Delta_m} := \mathbf{diag}\left(\frac{d}{dt} I_{m_0}, \mathbf{S}_1 I_{m_1}, \mathbf{S}_1^{-1} I_{m_{-1}}, \mathbf{S}_2 I_{m_2}, \mathbf{S}_2^{-1} I_{m_{-2}} \ldots, \mathbf{S}_L^{-1} I_{m_{-L}}\right). \tag{28}$$

IV. Well-Posedness, Stability, and Performance

There are three main considerations when analyzing a system: well-posedness, stability, and performance. In this section, these concepts are defined, and an LMI condition for establishing well-posedness, stability, and performance is presented.

A. Well-Posedness

Simply put, a system is well-posed if it is physically realizable. The following simple examples illustrate the concept of well-posedness. Consider the feedback interconnection in Fig. 5, where all signals are simply a function of time. Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be unity gain systems: $w_1(t) = v_1(t), w_2(t) = v_2(t)$. This interconnection is not well-posed because there do not exist solutions to the loop equations for all possible exogenous signals n_1 and n_2.

Now let $\mathbf{P}_1$ be a unity gain system, and let $\mathbf{P}_2$ be a linear time invariant system with transfer function $\mathrm{P}_2(\zeta) = 1 - 1/\zeta$. This interconnection is also not well-posed because the resulting transfer function from exogenous signal n_1 to interconnection signal v_1 is not proper, and in fact equal to ζ. There is thus differentiating action from one of the closed-loop system inputs to one of the closed-loop system outputs (all the closed-loop dependent variables are considered outputs: v_1, w_1, v_2, w_2). The reader is referred to [44] for an in-depth discussion of well-posedness.

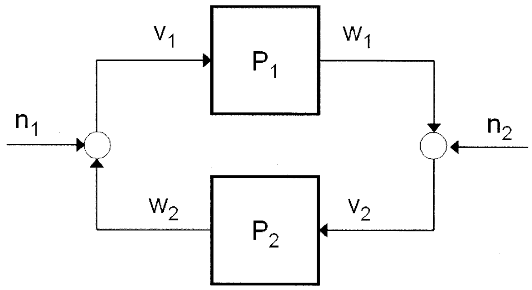

Fig. 5. Standard interconnection.

We can extend the definition of well-posedness in [44] to the systems considered in this paper.

Definition 3: Consider a system with signals n injected at the interconnection points

$$\begin{bmatrix} \dot{x}(t,\mathbf{s}) \\ w(t,\mathbf{s}) \\ z(t,\mathbf{s}) \end{bmatrix} = \begin{bmatrix} A_{\mathbf{TT}} & A_{\mathbf{TS}} & B_{\mathbf{T}} \\ A_{\mathbf{ST}} & A_{\mathbf{SS}} & B_{\mathbf{S}} \\ C_{\mathbf{T}} & C_{\mathbf{S}} & D \end{bmatrix} \begin{bmatrix} x(t,\mathbf{s}) \\ v(t,\mathbf{s}) \\ d(t,\mathbf{s}) \end{bmatrix} \quad (29)$$

$$(\mathbf{\Delta}_{\mathbf{S},\mathbf{m}} v(t))(\mathbf{s}) = w(t,\mathbf{s}) + n(t,\mathbf{s}). \quad (30)$$

A system is said to be *well-posed* if there exist strictly positive numbers δ and T such that for any inputs n and d which satisfy $\|n(t)\|_{\ell_2} \leq \delta, \|d(t)\|_{\ell_2} \leq \delta$ for all t in $[0,T]$, there exist unique signals x, v, w, and z which satisfy (29) and (30), with $x(t=0)=0$ and norm constraints

$$\|v(t)\|_{\ell_2} \leq 1, \|w(t)\|_{\ell_2} \leq 1, \|z(t)\|_{\ell_2} \leq 1 \quad \text{for all } t \in [0,T]. \quad (31)$$

The interpretation of well-posedness is equivalent to the standard one [44]: there must exist unique solutions to the system equations when signals are injected anywhere in the loop. In addition, on a sufficiently small time interval, the gain from signals injected anywhere in the loop to all system outputs must be bounded. This is depicted in Fig. 6 for a periodic interconnection in one spatial dimension. The proof of the following statement is found in the Appendix.

Proposition 1: A system is well-posed if and only if $(\mathbf{\Delta}_{\mathbf{S},\mathbf{m}} - A_{\mathbf{SS}})$ is invertible on ℓ_2.

We will always require that a given system is well-posed; conditions for establishing well-posedness are presented in Section IV-D. One method for ensuring system well-posedness is to simply require that no direct feed-through terms exist in an interconnection ($A_{\mathbf{SS}} = 0$), since $\mathbf{\Delta}_{\mathbf{S},\mathbf{m}}$ is always invertible. The physical interpretation of this requirement is that information transfer among the subsystems is bandwidth limited.

B. Stability

For a well-posed system, operators $\mathbf{A}, \mathbf{B}, \mathbf{C}$, and $\mathbf{D}$ are bounded, and we may readily write down the solution to (18) for some intitial condition $x_0 \in \ell_2$:

$$x(t) = \exp(\mathbf{A}t)x_0 + \int_0^t \exp(\mathbf{A}(t-\tau))\mathbf{B}d(\tau)\, d\tau \quad (32)$$

where $\exp(\mathbf{A}t)$ is the continuous semigroup defined by

$$\exp(\mathbf{A}t) := \sum_{n=0}^{\infty} \frac{(\mathbf{A}t)^n}{n!} \quad (33)$$

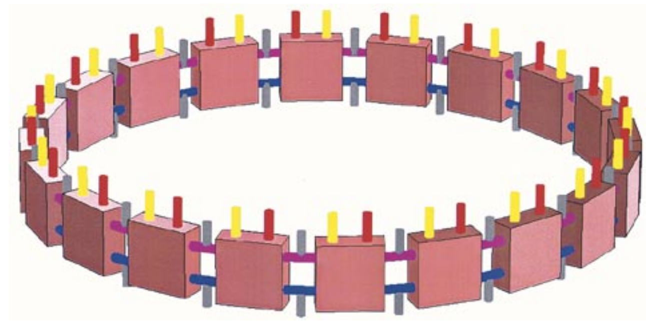

Fig. 6. Test for well-posedness, one spatial dimension, periodic interconnection.

the reader is referred to [10] and [4] for details. For the reader not familiar with semigroup theory, the key point is that the boundedness of $\mathbf{A}, \mathbf{B}, \mathbf{C}$, and $\mathbf{D}$ allows us to formally treat these systems analogous to their finite dimensional counterparts. Also note that $\mathbf{A}, \mathbf{B}, \mathbf{C}$, and $\mathbf{D}$ are bounded since the underlying spatial dynamics being considered are discrete; this should be compared with spatially continuous systems which typically have unbounded system operators [10].

A system $\mathcal{M}$ is said to be *stable* if there exist α and β greater than zero such that for all t

$$\|\exp(\mathbf{A}t)\|_{\ell_2} \leq \alpha \exp(-\beta t). \quad (34)$$

This is often referred to as exponential stability [10].

C. Performance

When a system $\mathcal{M}$ is stable, we define $\mathbf{M}$ to be the operator which maps $d \in \mathcal{L}_2$ to $z \in \mathcal{L}_2$ for zero initial conditions. The induced $\mathcal{L}_2$ gain of a bounded system $\mathcal{M}$ is denoted $\|\mathbf{M}\|_{\mathcal{L}_2}$. When this gain is less than one, system $\mathcal{M}$ is said to be *contractive*. Contractiveness is the performance measure used throughout the paper; in particular, when we consider control design in Section V, the objective will be to design a controller which renders the closed-loop system contractive.

D. Analysis Condition for Well-Posedness, Stability, and Performance

Given a system $\mathcal{M} = \{A, B, C, D, \mathbf{m}\}$, the equations governing the evolution of the system can be partitioned according to the spatial and temporal components of $\mathbf{\Delta}_{\mathbf{m}}$, as per (17). The data can further be partitioned to reflect the structure of $\mathbf{\Delta}_{\mathbf{S},\mathbf{m}}$:

$$A_{\mathbf{SS}} =: \begin{bmatrix} A_{\mathbf{SS}_{1,1}} & A_{\mathbf{SS}_{1,-1}} & \cdots & A_{\mathbf{SS}_{1,-L}} \\ A_{\mathbf{SS}_{-1,1}} & A_{\mathbf{SS}_{-1,-1}} & \cdots & A_{\mathbf{SS}_{-1,-L}} \\ & & \ddots & \\ A_{\mathbf{SS}_{-L,1}} & A_{\mathbf{SS}_{-L,-1}} & \cdots & A_{\mathbf{SS}_{-L,-L}} \end{bmatrix}$$

$$A_{\mathbf{ST}} =: \begin{bmatrix} A_{\mathbf{ST}_1} \\ A_{\mathbf{ST}_{-1}} \\ \vdots \\ A_{\mathbf{ST}_{-L}} \end{bmatrix} \quad B_{\mathbf{S}} =: \begin{bmatrix} B_{\mathbf{S}_1} \\ B_{\mathbf{S}_{-1}} \\ \vdots \\ B_{\mathbf{S}_{-L}} \end{bmatrix} \quad (35)$$

$$A_{\mathbf{TS}} =: [\, A_{\mathbf{TS}_1} \quad A_{\mathbf{TS}_{-1}} \quad \cdots \quad A_{\mathbf{TS}_{-L}} \,]$$

$$C_{\mathbf{S}} =: [\, C_{\mathbf{S}_1} \quad C_{\mathbf{S}_{-1}} \quad \cdots \quad C_{\mathbf{S}_{-L}} \,]. \quad (36)$$

Define the matrices as shown in

$$A_{\mathbf{SS}}^{+} := \begin{bmatrix} A_{\mathbf{SS}_{1,1}} & A_{\mathbf{SS}_{1,-1}} & \cdots & A_{\mathbf{SS}_{1,L}} & A_{\mathbf{SS}_{1,-L}} \\ 0 & I & \cdots & 0 & 0 \\ & & \ddots & & \\ A_{\mathbf{SS}_{L,1}} & A_{\mathbf{SS}_{L,-1}} & \cdots & A_{\mathbf{SS}_{L,L}} & A_{\mathbf{SS}_{L,-L}} \\ 0 & 0 & \cdots & 0 & I \end{bmatrix}$$

$$A_{\mathbf{ST}}^{+} := \begin{bmatrix} A_{\mathbf{ST}_1} \\ 0 \\ \vdots \\ A_{\mathbf{ST}_L} \\ 0 \end{bmatrix} \quad B_{\mathbf{S}}^{+} := \begin{bmatrix} B_{\mathbf{S}_1} \\ 0 \\ \vdots \\ B_{\mathbf{S}_L} \\ 0 \end{bmatrix} \tag{37}$$

$$A_{\mathbf{SS}}^{-} := \begin{bmatrix} I & 0 & \cdots & 0 & 0 \\ A_{\mathbf{SS}_{-1,1}} & A_{\mathbf{SS}_{-1,-1}} & \cdots & A_{\mathbf{SS}_{-1,L}} & A_{\mathbf{SS}_{-1,-L}} \\ & & \ddots & & \\ 0 & 0 & \cdots & I & 0 \\ A_{\mathbf{SS}_{-L,1}} & A_{\mathbf{SS}_{-L,-1}} & \cdots & A_{\mathbf{SS}_{-L,L}} & A_{\mathbf{SS}_{-L,-L}} \end{bmatrix}$$

$$A_{\mathbf{ST}}^{-} := \begin{bmatrix} 0 \\ A_{\mathbf{ST}_{-1}} \\ \vdots \\ 0 \\ A_{\mathbf{ST}_{-L}} \end{bmatrix} \quad B_{\mathbf{S}}^{-} := \begin{bmatrix} 0 \\ B_{\mathbf{S}_{-1}} \\ \vdots \\ 0 \\ B_{\mathbf{S}_{-L}} \end{bmatrix} \tag{38}$$

$$\begin{aligned} A_{\mathbf{TS}}^{+} &:= [A_{\mathbf{TS}_1} \quad 0 \quad \cdots \quad A_{\mathbf{TS}_L} \quad 0] \\ A_{\mathbf{TS}}^{-} &:= [0 \quad A_{\mathbf{TS}_{-1}} \quad \cdots \quad 0 \quad A_{\mathbf{TS}_{-L}}]. \end{aligned} \tag{39}$$

Define the following sets of scaling matrices:

$$\mathcal{X}_{\mathbf{T}} := \left\{ X_{\mathbf{T}} \in \mathbb{R}_{\mathbf{S}}^{m_0 \times m_0} : X_{\mathbf{T}} > 0 \right\} \tag{40}$$

$$\mathcal{X}_{\mathbf{S}} := \Big\{ X_{\mathbf{S}} = \mathbf{diag}\,(X_{\mathbf{S}_1}, X_{\mathbf{S}_2}, \ldots, X_{\mathbf{S}_L}) : X_{\mathbf{S}_i} \in \mathbb{R}_{\mathbf{S}}^{(m_i+m_{-i})\times(m_i+m_{-i})} \Big\}. \tag{41}$$

The following result allows us to check the well-posedness, stability, and performance of a system via an LMI. The proof may be found in the Appendix.

Theorem 1: A system $\mathcal{M} = \{A, B, C, D, \mathbf{m}\}$ is well-posed, stable, and contractive if there exist $X_{\mathbf{T}}$ in $\mathcal{X}_{\mathbf{T}}$ and $X_{\mathbf{S}}$ in $\mathcal{X}_{\mathbf{S}}$ such that $J < 0$, where

$$\begin{aligned} J := &\begin{bmatrix} I & 0 & 0 \\ A_{\mathbf{ST}}^{-} & A_{\mathbf{SS}}^{-} & B_{\mathbf{S}}^{-} \\ 0 & 0 & I \end{bmatrix}^{*} \\ &\times \begin{bmatrix} A_{\mathbf{TT}}^{*} X_{\mathbf{T}} + X_{\mathbf{T}} A_{\mathbf{TT}} & X_{\mathbf{T}} A_{\mathbf{TS}}^{+} & X_{\mathbf{T}} B_{\mathbf{T}} \\ (A_{\mathbf{TS}}^{+})^{*} X_{\mathbf{T}} & -X_{\mathbf{S}} & 0 \\ B_{\mathbf{T}}^{*} X_{\mathbf{T}} & 0 & -I \end{bmatrix} \\ &\times \begin{bmatrix} I & 0 & 0 \\ A_{\mathbf{ST}}^{-} & A_{\mathbf{SS}}^{-} & B_{\mathbf{S}}^{-} \\ 0 & 0 & I \end{bmatrix} + \begin{bmatrix} I & 0 & 0 \\ A_{\mathbf{ST}}^{+} & A_{\mathbf{SS}}^{+} & B_{\mathbf{S}}^{+} \\ C_{\mathbf{T}} & C_{\mathbf{S}} & D \end{bmatrix}^{*} \\ &\times \begin{bmatrix} 0 & X_{\mathbf{T}} A_{\mathbf{TS}}^{-} & 0 \\ (A_{\mathbf{TS}}^{-})^{*} X_{\mathbf{T}} & X_{\mathbf{S}} & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ A_{\mathbf{ST}}^{+} & A_{\mathbf{SS}}^{+} & B_{\mathbf{S}}^{+} \\ C_{\mathbf{T}} & C_{\mathbf{S}} & D \end{bmatrix}. \end{aligned} \tag{42}$$

Remarks:

- The analysis condition is valid for both periodic and infinite interconnections, and is independent of the number of blocks in a periodic interconnection. The size of the resulting LMI is only a function of the size of the basic building block used to describe the interconnection.
- The condition $J < 0$ may be conservative in capturing the stability and performance requirements. This is intimately tied to the fact that μ, the structured singular value, is generally not equal to its upper bound [33].
- In the absence of spatial dynamics (no operator $\mathbf{\Delta}_{\mathbf{S},\mathbf{m}}$), condition $J < 0$ simply reduces to the KYP Lemma (see [36], for example). Also note that in the absence of temporal dynamics (no state x), and in one spatial dimension with only forward shifts ($\mathbf{\Delta}_{\mathbf{S},\mathbf{m}} = \mathbf{S}I$), the above reduces to the discrete time version of the KYP lemma, with the exception of the missing constraint $X_{\mathbf{S}} > 0$; this constraint would impose spatial causality (see [14], for example), which is not a requirement for the types of systems considered in this paper.

By eliminating all inputs, outputs, and temporal dynamics, one may readily extract an LMI condition for establishing system well-posedness.

Corollary 1: A system $\mathcal{M} = \{A, B, C, D, \mathbf{m}\}$ is well-posed if there exists $X_{\mathbf{S}}$ in $\mathcal{X}_{\mathbf{S}}$ such that

$$\left(A_{\mathbf{SS}}^{+}\right)^{*} X_{\mathbf{S}} A_{\mathbf{SS}}^{+} - (A_{\mathbf{SS}}^{-})^{*} X_{\mathbf{S}} A_{\mathbf{SS}}^{-} < 0 \tag{43}$$

It can readily be shown that this condition is also necessary when $L = 1$ (one spatial dimension).

For a given system, the condition in Theorem 1 yields a tractable method for checking the well-posedness, stability, and performance of a system, since it is an LMI in the decision variables $X_{\mathbf{T}}$ and $X_{\mathbf{S}}$. In what follows, we provide an alternate test for well-posedness, stability, and performance based on the condition in Theorem 1 which will be used directly for controller synthesis in Section V. We will first require the use of the following matrix transformation.

Definition 4: Given a system $\mathcal{M} = \{A, B, C, D, \mathbf{m}\}$, where $I - A_{\mathbf{SS}}$ is assumed to be invertible, let H be the following matrix:

$$H = \begin{bmatrix} I_{m_1} & 0 & \cdots & 0 \\ 0 & -I_{m_{-1}} & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & -I_{m_{-L}} \end{bmatrix}. \tag{44}$$

Define function f_{D2C} as

$$f_{D2C}(\mathcal{M}) := \bar{\mathcal{M}} = \{\bar{A}, \bar{B}, \bar{C}, \bar{D}, \bar{\mathbf{m}}\} \tag{45}$$

where

$$\bar{\mathbf{m}} := (m_0, m_1 + m_{-1}, 0, \ldots, m_L + m_{-L}, 0) \tag{46}$$

$$\bar{A}_{\mathbf{SS}} := -(I + A_{\mathbf{SS}})(I - A_{\mathbf{SS}})^{-1} H \tag{47}$$

$$[\bar{A}_{\mathbf{ST}} \quad \bar{B}_{\mathbf{S}}] := \sqrt{2}(I - A_{\mathbf{SS}})^{-1}[A_{\mathbf{ST}} \quad B_{\mathbf{S}}] \tag{48}$$

$$\begin{bmatrix} \bar{A}_{\mathbf{TS}} \\ \bar{C}_{\mathbf{S}} \end{bmatrix} := -\sqrt{2} \begin{bmatrix} A_{\mathbf{TS}} \\ C_{\mathbf{S}} \end{bmatrix} (I - A_{\mathbf{SS}})^{-1} H \tag{49}$$

$$\begin{aligned} \begin{bmatrix} \bar{A}_{\mathbf{TT}} & \bar{B}_{\mathbf{T}} \\ \bar{C}_{\mathbf{T}} & \bar{D} \end{bmatrix} := &\begin{bmatrix} A_{\mathbf{TT}} & B_{\mathbf{T}} \\ C_{\mathbf{T}} & D \end{bmatrix} + \begin{bmatrix} A_{\mathbf{TS}} \\ C_{\mathbf{S}} \end{bmatrix} \\ &\times (I - A_{\mathbf{SS}})^{-1}[A_{\mathbf{ST}} \quad B_{\mathbf{S}}]. \end{aligned} \tag{50}$$

This transformation is a modified bilinear transformation. The subscript "D2C" is in fact used to suggest that it is very similar to the standard bilinear transformation used to convert a discrete time $\mathcal{H}_\infty$ problem to a continuous time $\mathcal{H}_\infty$ problem [2].

Theorem 2: Given a system $\mathcal{M} = \{A, B, C, D, \mathbf{m}\}$ and scaling matrices $X_{\mathbf{T}}$ in $\mathcal{X}_{\mathbf{T}}$ and $X_{\mathbf{S}}$ in $\mathcal{X}_{\mathbf{S}}$, define J as per (42). The following two conditions are equivalent:

I) $J < 0$.

II) The following two conditions are satisfied:

1) $I - A_{\mathbf{SS}}$ is invertible;
2) the following inequality is satisfied:

$$\begin{bmatrix} \bar{A}^*X + X\bar{A} & X\bar{B} & \bar{C}^* \\ \bar{B}^*X & -I & \bar{D}^* \\ \bar{C} & \bar{D} & -I \end{bmatrix} < 0 \tag{51}$$

where $X := \mathbf{diag}(X_{\mathbf{T}}, X_{\mathbf{S}})$ and $\bar{\mathcal{M}} = \{\bar{A}, \bar{B}, \bar{C}, \bar{D}, \bar{\mathbf{m}}\} = f_{D2C}(\mathcal{M})$.

The proof may be found in the Appendix. Note that the matrix on the left-hand side of (51) is affine in the system data $\bar{A}, \bar{B}, \bar{C}$, and $\bar{D}$, which is not the case for J; as we shall see in Section V, this equivalent check for well-posedness, stability, and performance is instrumental in obtaining convex synthesis conditions.

Note that the condition in (51) is very similar to the continuous time version of the KYP Lemma. The only differences are that X is structured and not necessarily positive definite.

V. Control Design Problem

For control design, the basic building block is augmented to include sensor and actuator variables, as depicted in Fig. 7 for one spatial dimension. In particular, let $\mathcal{M}^{\mathrm{G}} = \{A^{\mathrm{G}}, B^{\mathrm{G}}, C^{\mathrm{G}}, D^{\mathrm{G}}, \mathbf{m}^{\mathrm{G}}\}$ be the given open-loop plant

$$\begin{bmatrix} (\boldsymbol{\Delta}_{\mathbf{m}^{\mathrm{G}}} r^{\mathrm{G}})(t,\mathbf{s}) \\ z(t,\mathbf{s}) \\ y(t,\mathbf{s}) \end{bmatrix} = \begin{bmatrix} A^{\mathrm{G}} & B^{\mathrm{G}}_{\mathbf{d}} & B^{\mathrm{G}}_{\mathbf{u}} \\ C^{\mathrm{G}}_{\mathbf{z}} & D^{\mathrm{G}}_{\mathbf{zd}} & D^{\mathrm{G}}_{\mathbf{zu}} \\ C^{\mathrm{G}}_{\mathbf{y}} & D^{\mathrm{G}}_{\mathbf{yd}} & D^{\mathrm{G}}_{\mathbf{yu}} \end{bmatrix} \begin{bmatrix} r^{\mathrm{G}}(t,\mathbf{s}) \\ d(t,\mathbf{s}) \\ u(t,\mathbf{s}) \end{bmatrix} \tag{52}$$

where $B^{\mathrm{G}}, C^{\mathrm{G}}$, and D^{G} have been partitioned as to be consistent with the partition of the inputs into d and u, and the outputs into z and y. Signals d are the exogenous disturbances, u the control signals, z the error signals which must be kept small, and y the sensor signals.

The control system $\mathcal{M}^{\mathrm{K}}$ to be designed will have y as its inputs and u as its outputs. We restrict ourselves to controllers that have the same structure as the plant's. For example, the basic building block for the controller, in one spatial dimension, is depicted in Fig. 8. The resulting closed-loop system, for periodic interconnections in one spatial dimension, is depicted in Fig. 9.

In particular, the control design objective is to construct a system $\mathcal{M}^{\mathrm{K}} = \{A^{\mathrm{K}}, B^{\mathrm{K}}, C^{\mathrm{K}}, D^{\mathrm{K}}, \mathbf{m}^{\mathrm{K}}\}$

$$\begin{bmatrix} (\boldsymbol{\Delta}_{\mathbf{m}^{\mathrm{K}}} r^{\mathrm{K}})(t,\mathbf{s}) \\ u(t,\mathbf{s}) \end{bmatrix} = \begin{bmatrix} A^{\mathrm{K}} & B^{\mathrm{K}} \\ C^{\mathrm{K}} & D^{\mathrm{K}} \end{bmatrix} \begin{bmatrix} r^{\mathrm{K}}(t,\mathbf{s}) \\ y(t,\mathbf{s}) \end{bmatrix} \tag{53}$$

such that the closed-loop system $\mathcal{M}$ is well-posed, stable, and contractive. Note that in the absence of spatial dynamics (no interconnection variables), this simply reduces to the standard $\mathcal{H}_\infty$ design problem [18].

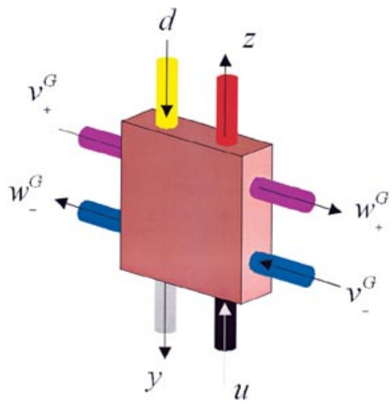

Fig. 7. Basic building block for control design, one spatial dimension.

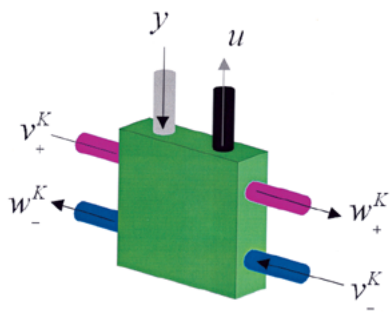

Fig. 8. Basic building block for controller, one spatial dimension.

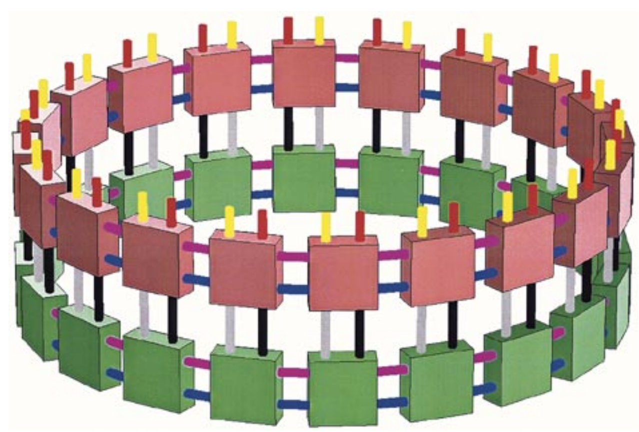

Fig. 9. Closed-loop system, periodic interconnection, one spatial dimension.

We may readily construct the data $\{A, B, C, D, \mathbf{m}\}$ which define the closed-loop system $\mathcal{M}$. Assuming that $(I - D^{\mathrm{G}}_{\mathbf{yd}} D^{\mathrm{K}})$ is invertible, y and u can be eliminated from (52) and (53) to yield

$$\begin{bmatrix} \begin{bmatrix} (\boldsymbol{\Delta}_{\mathbf{m}^{\mathrm{G}}} r^{G})(t,\mathbf{s}) \\ (\boldsymbol{\Delta}_{\mathbf{m}^{\mathrm{K}}} r^{\mathrm{K}})(t,\mathbf{s}) \end{bmatrix} \\ z(t,\mathbf{s}) \end{bmatrix} = \begin{bmatrix} A^{\mathrm{C}} & B^{\mathrm{C}} \\ C^{\mathrm{C}} & D^{\mathrm{C}} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} r^{\mathrm{G}}(t,\mathbf{s}) \\ r^{\mathrm{K}}(t,\mathbf{s}) \end{bmatrix} \\ d(t,\mathbf{s}) \end{bmatrix} \tag{54}$$

where matrices $A^{\mathrm{C}}, B^{\mathrm{C}}, C^{\mathrm{C}}$, and D^{C} can readily be constructed from the matrices in $\mathcal{M}^{\mathrm{G}}$ and $\mathcal{M}^{\mathrm{K}}$; the details are omitted, since we will not be exploiting this dependence in the text. The closed-loop system equations in (54) are not in the standard form given in (23). In particular, $(r^{\mathrm{G}}, r^{\mathrm{K}}) = (r^{\mathrm{G}}_{\mathbf{T}}, r^{\mathrm{G}}_{\mathbf{S}}, r^{\mathrm{K}}_{\mathbf{T}}, r^{\mathrm{K}}_{\mathbf{S}})$, and thus the temporal and spatial variables are not grouped together as they are in (23). Define permutation matrix P as follows:

$$P := \begin{bmatrix} P^{\mathrm{G}}_{\mathbf{T}} & 0 & P^{\mathrm{K}}_{\mathbf{T}} & 0 \\ 0 & P^{\mathrm{G}}_{\mathbf{S}} & 0 & P^{\mathrm{K}}_{\mathbf{S}} \end{bmatrix} \tag{55}$$

where

$$P^{\mathrm{G}}_{\mathbf{T}} := \begin{bmatrix} I_{m_0^{\mathrm{G}}} \\ 0 \end{bmatrix} \quad P^{\mathrm{K}}_{\mathbf{T}} := \begin{bmatrix} 0 \\ I_{m_0^{\mathrm{K}}} \end{bmatrix} \tag{56}$$

$$P^{\mathrm{G}}_{\mathbf{S}} := \begin{bmatrix} I_{m_1^{\mathrm{G}}} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & I_{m_{-1}^{\mathrm{G}}} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ & & \vdots & \\ 0 & 0 & \cdots & I_{m_{-L}^{\mathrm{G}}} \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$P^{\mathrm{K}}_{\mathbf{S}} := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ I_{m_1^{\mathrm{K}}} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & I_{m_{-1}^{\mathrm{K}}} & \cdots & 0 \\ & & \vdots & \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & I_{m_{-L}^{\mathrm{K}}} \end{bmatrix}. \tag{57}$$

Note that $PP^* = P^*P = I$, and that

$$P\mathbf{diag}(\mathbf{\Delta}_{\mathbf{m}^{\mathrm{G}}}, \mathbf{\Delta}_{\mathbf{m}^{\mathrm{K}}})P^* = \mathbf{\Delta}_{\mathbf{m}} \tag{58}$$

where $\mathbf{m} := \mathbf{m}^{\mathrm{G}} + \mathbf{m}^{\mathrm{K}}$, i.e., $m_i := m_i^{\mathrm{G}} + m_i^{\mathrm{K}}$. The closed-loop system is thus $\mathcal{M} = \{A, B, C, D, \mathbf{m}\}$, where

$$A := PA^{\mathrm{C}}P^* \quad B := PB^{\mathrm{C}} \quad C := C^{\mathrm{C}}P^* \quad D := D^{\mathrm{C}} \tag{59}$$

as per (23), where $r = P(r^{\mathrm{G}}, r^{\mathrm{K}})$. Note that $\{A, B, C, D, \mathbf{m}\}$ are unique and well defined as long as $(I - D^{\mathrm{G}}_{\mathbf{yd}}D^{\mathrm{K}})$ is invertible. It can readily be shown that when this matrix is not invertible, the closed-loop system is not well-posed. Define the function f_{IC} which generates the data in $\mathcal{M}$ from $\mathcal{M}^{\mathrm{K}}$ and $\mathcal{M}^{\mathrm{G}}$ as follows: $\mathcal{M} =: f_{\mathrm{IC}}(\mathcal{M}^{\mathrm{G}}, \mathcal{M}^{\mathrm{K}})$. We can then formalize the problem formulation as follows.

Problem Formulation*:* *Given a system $\mathcal{M}^G$, find a system $\mathcal{M}^K$ such that $(I - D^G_{\mathbf{yd}}D^K)$ is invertible, and the system defined by $\mathcal{M} = f_{\mathrm{IC}}(\mathcal{M}^G, \mathcal{M}^K)$ is well-posed, stable, and contractive.*

We will solve the control design problem by constructing $\mathcal{M}^{\mathrm{K}}$ *and* scaling matrices $X_{\mathbf{T}}$ in $\mathcal{X}_{\mathbf{T}}$ and $X_{\mathbf{S}}$ in $\mathcal{X}_{\mathbf{S}}$ such that the inequality in (51) is satisfied. Note that even though the inequality in (51) is affine in $\bar{A}, \bar{B}, \bar{C}$, and $\bar{D}$, and affine in X, and thus independently convex in $\bar{\mathcal{M}}$ and X, it is not jointly convex in $\bar{\mathcal{M}}$ and X. As we shall see, the tools in [32], [24], and [1] can be brought to bear on the inequality in (51) and the problem convexified.

A complication with the inequality in (51) is that the controller data $\mathcal{M}^{\mathrm{K}}$ appears in the inequality *after* the plant and controller data are jointly transformed via $f_{\mathrm{D2C}} : \bar{\mathcal{M}} = f_{\mathrm{D2C}}(f_{\mathrm{IC}}(\mathcal{M}^{\mathrm{G}}, \mathcal{M}^{\mathrm{K}}))$. The following result, however, states that the order in which the transformation f_{D2C} and the interconnection f_{IC} are applied can be interchanged; the proof follows from straight forward matrix manipulations, and is thus omitted.

Proposition 2: Given $\mathcal{M}^G$ and $\mathcal{M}^K$, assume that $(I - A^G_{\mathbf{SS}})$ and $(I - A^K_{\mathbf{SS}})$ are invertible, and define $\bar{\mathcal{M}}^G := f_{D2C}(\mathcal{M}^G)$ and $\bar{\mathcal{M}}^K := f_{D2C}(\mathcal{M}^K)$. If $(I - D^G_{\mathbf{yu}}D^K)$ and $(I - \bar{D}^G_{\mathbf{yu}}\bar{D}^K)$ are invertible, then

$$f_{D2C}(f_{IC}(\mathcal{M}^G, \mathcal{M}^K)) = f_{\mathrm{IC}}(\bar{\mathcal{M}}^G, \bar{\mathcal{M}}^K). \tag{60}$$

We may thus first transform $\mathcal{M}^{\mathrm{G}}$ to yield $\bar{\mathcal{M}}^{\mathrm{G}} = f_{\mathrm{D2C}}(\mathcal{M}^{\mathrm{G}})$, use the condition in (51) to find a suitable $\bar{\mathcal{M}}^{\mathrm{K}}$, and upon finding such an $\bar{\mathcal{M}}^{\mathrm{K}}$, find an $\mathcal{M}^{\mathrm{K}}$ such that $\bar{\mathcal{M}}^{\mathrm{K}} = f_{\mathrm{D2C}}(\mathcal{M}^{\mathrm{K}})$. The details of constructing a suitable $\mathcal{M}^{\mathrm{K}}$ from $\bar{\mathcal{M}}^{\mathrm{K}}$ are deferred to Section VI.

In order to perform transformation f_{D2C}, matrices $(I - A^{\mathrm{G}}_{\mathbf{SS}})$ and $(I - A^{\mathrm{K}}_{\mathbf{SS}})$ must be invertible. This is in fact a natural assumption on the plant and controller: If $(I - A^{\mathrm{G}}_{\mathbf{SS}})$ is not invertible, the well-posedness assumption is violated; similarly for $(I - A^{\mathrm{K}}_{\mathbf{SS}})$. This is captured by Lemma 1. As discussed in Section IV-A, it is reasonable to assume that the plant and controller are governed by well-posed systems of equations, justifying this assumption.

Lemma 1: If $(\mathbf{\Delta}_{\mathbf{S},\mathbf{m}} - A_{\mathbf{SS}})$ is invertible on ℓ_2, then $(I - A_{\mathbf{SS}})$ is invertible.

The proof may be found in the Appendix. Note that we must also assume that $(I - D^{\mathrm{G}}_{\mathbf{yu}}D^{\mathrm{K}})$ and $(I - \bar{D}^{\mathrm{G}}_{\mathbf{yu}}\bar{D}^{\mathrm{K}})$ are invertible in Proposition 2. This assumption is not restrictive, however, since the problem formulation requires $(I - D^{\mathrm{G}}_{\mathbf{yu}}D^{\mathrm{K}})$ to be invertible for well-posedness, and $\bar{D}^{\mathrm{K}}$ can always be perturbed by a small amount if a candidate $\bar{\mathcal{M}}^{\mathrm{K}}$ results in a singular $(I - \bar{D}^{\mathrm{G}}_{\mathbf{yu}}\bar{D}^{\mathrm{K}})$.

We are now in a position to apply the tools in [24] to obtain LMI conditions for controller synthesis. The development is virtually the same as that in [24], with two differences.

1) Scaling matrix X is not positive definite. This will affect the coupling condition which typically arises in the LMI formulation of $\mathcal{H}_\infty$ optimization.
2) Scaling matrix X is structured. This is analogous to the gain scheduling results in [32] and [1].

For a given $\mathcal{M}^{\mathrm{G}}$, let $\mathcal{M}^{\mathrm{K}}$ be a candidate controller. Define $\bar{\mathcal{M}}^{\mathrm{G}} = f_{\mathrm{D2C}}(\mathcal{M}^{\mathrm{G}})$ and $\bar{\mathcal{M}}^{\mathrm{K}} = f_{\mathrm{D2C}}(\mathcal{M}^{\mathrm{K}})$. Assume, without loss of generality, that $\bar{D}^{\mathrm{G}}_{\mathbf{yu}} = 0$; this is a standard approach known as loop shifting [38]. In particular, if $\bar{\mathcal{M}}^{\mathrm{K}}$ is designed under this assumption, it can be mapped via the transformation for nonzero $\bar{D}^{\mathrm{G}}_{\mathbf{yu}}$ (see [38] for details) shown in (61) at the bottom of the page. We may thus design the controller assuming

$$\begin{bmatrix} \bar{A}^{\mathrm{K}} & \bar{B}^{\mathrm{K}} \\ \bar{C}^{\mathrm{K}} & \bar{D}^{\mathrm{K}} \end{bmatrix} \longrightarrow \begin{bmatrix} \bar{A}^{\mathrm{K}} - \bar{B}^{\mathrm{K}}\bar{D}^{\mathrm{G}}_{\mathbf{yu}}(I + \bar{D}^{\mathrm{K}}\bar{D}^{\mathrm{G}}_{\mathbf{yu}})^{-1}\bar{C}^{\mathrm{K}} & \bar{B}^{\mathrm{K}}(I + \bar{D}^{\mathrm{G}}_{\mathbf{yu}}\bar{D}^{\mathrm{K}})^{-1} \\ (I + \bar{D}^{\mathrm{K}}\bar{D}^{\mathrm{G}}_{\mathbf{yu}})^{-1}\bar{C}^{\mathrm{K}} & (I + \bar{D}^{\mathrm{K}}\bar{D}^{\mathrm{G}}_{\mathbf{yu}})^{-1}\bar{D}^{\mathrm{K}} \end{bmatrix}. \tag{61}$$

that $\bar{D}^{\rm G}_{\bf yu} = 0$, and then apply the transformation to yield the required controller equations.

Let $\bar{\mathcal{M}} = f_{\rm IC}(\bar{\mathcal{M}}^{\rm G}, \bar{\mathcal{M}}^{\rm K})$. Let P be the permutation matrix in (58). Then it can readily be shown that

$$\begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} = \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{A}^{\rm C} & \bar{B}^{\rm C} \\ \bar{C}^{\rm C} & \bar{D}^{\rm C} \end{bmatrix} \begin{bmatrix} P^* & 0 \\ 0 & I \end{bmatrix} \tag{62}$$

where

$$\begin{bmatrix} \bar{A}^{\rm C} & \bar{B}^{\rm C} \\ \bar{C}^{\rm C} & \bar{D}^{\rm C} \end{bmatrix} := \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} + \begin{bmatrix} \mathcal{B} \\ \mathcal{D}_{12} \end{bmatrix} \Theta [\mathcal{C} \quad \mathcal{D}_{21}] \tag{63}$$

$$A_0 := \begin{bmatrix} \bar{A}^{\rm G} & 0 \\ 0 & 0 \end{bmatrix} \quad B_0 := \begin{bmatrix} \bar{B}^{\rm G}_{\bullet \bf d} \\ 0 \end{bmatrix}$$

$$C_0 := [\bar{C}^{\rm G}_{\bf z\bullet} \quad 0] \quad D_0 := \bar{D}^{\rm G}_{\bf zd} \tag{64}$$

$$\mathcal{B} := \begin{bmatrix} 0 & \bar{B}^{\rm G}_{\bf u\bullet} \\ I & 0 \end{bmatrix} \quad \mathcal{C} := \begin{bmatrix} 0 & I \\ \bar{C}^{\rm G}_{\bf y\bullet} & 0 \end{bmatrix}$$

$$\mathcal{D}_{12} := [0 \quad \bar{D}^{\rm G}_{\bf zu}] \quad \mathcal{D}_{21} := \begin{bmatrix} 0 \\ \bar{D}^{\rm G}_{\bf yd} \end{bmatrix} \tag{65}$$

$$\Theta := \begin{bmatrix} \bar{A}^{\rm K} & \bar{B}^{\rm K} \\ \bar{C}^{\rm K} & \bar{D}^{\rm K} \end{bmatrix}. \tag{66}$$

Modulo notational differences and permutation matrix P, these expressions are identical to [24, eqs. (7), (8), and (9)]. When there are no spatial operators $\mathbf{\Delta}_{\bf S,m}$, permutation matrix P is equal to the identity and the expressions are equivalent to those in [24].

We can express the inequality in (51) as

$$\begin{bmatrix} (\bar{A}^{\rm C})^* \bar{X} + \bar{X} \bar{A}^{\rm C} & \bar{X} \bar{B}^{\rm C} & (\bar{C}^{\rm C})^* \\ (\bar{B}^{\rm C})^* \bar{X} & -I & (\bar{D}^{\rm C})^* \\ \bar{C}^{\rm C} & \bar{D}^{\rm C} & -I \end{bmatrix} < 0 \tag{67}$$

where $\bar{X} := P^* X P$. Recall the structure of scaling matrix$X$

$$X := \mathbf{diag}(X_{\bf T}, X_{\bf S}) \quad X_{\bf S} := \mathbf{diag}(X_{{\bf S},1}, \ldots, X_{{\bf S},L}) \tag{68}$$

$$X_{{\bf S},i} := \begin{bmatrix} X^{\rm G}_{{\bf S},i} & X^{\rm GK}_{{\bf S},i} \\ \left(X^{\rm GK}_{{\bf S},i}\right)^* & X^{\rm K}_{{\bf S},i} \end{bmatrix} \quad X_{\bf T} := \begin{bmatrix} X^{\rm G}_{\bf T} & X^{\rm GK}_{\bf T} \\ \left(X^{\rm GK}_{\bf T}\right)^* & X^{\rm K}_{\bf T} \end{bmatrix} \tag{69}$$

$$X^{\rm G}_{{\bf S},i} \in \mathbb{R}_{\bf S}^{\bar{m}_i^{\rm G} \times \bar{m}_i^{\rm G}}, X^{\rm K}_{{\bf S},i} \in \mathbb{R}_{\bf S}^{\bar{m}_i^{\rm K} \times \bar{m}_i^{\rm K}}, X^{\rm GK}_{{\bf S},i} \in \mathbb{R}^{\bar{m}_i^{\rm G} \times \bar{m}_i^{\rm K}} \tag{70}$$

$$X^{\rm G}_{\bf T} \in \mathbb{R}_{\bf S}^{\bar{m}_0^{\rm G} \times \bar{m}_0^{\rm G}}, X^{\rm K}_{\bf T} \in \mathbb{R}_{\bf S}^{\bar{m}_0^{\rm K} \times \bar{m}_0^{\rm K}}, X^{\rm GK}_{\bf T} \in \mathbb{R}^{\bar{m}_0^{\rm G} \times \bar{m}_0^{\rm K}}, \quad X_{\bf T} > 0. \tag{71}$$

Scaling matrix $\bar{X}$ inherits the following structure:

$$\bar{X} := \begin{bmatrix} X^{\rm G} & X^{\rm GK} \\ (X^{\rm GK})^* & X^{\rm K} \end{bmatrix} \tag{72}$$

where

$$X^{\rm G} := \mathbf{diag}\left(X^{\rm G}_{\bf T}, X^{\rm G}_{{\bf S},1}, \ldots, X^{\rm G}_{{\bf S},L}\right)$$
$$X^{\rm K} := \mathbf{diag}\left(X^{\rm K}_{\bf T}, X^{\rm K}_{{\bf S},1}, \ldots, X^{\rm K}_{{\bf S},L}\right)$$
$$X^{\rm GK} := \mathbf{diag}\left(X^{\rm GK}_{\bf T}, X^{\rm GK}_{{\bf S},1}, \ldots, X^{\rm GK}_{{\bf S},L}\right). \tag{73}$$

Define the following sets of scaling matrices:

$$\mathcal{X}^{\rm G} := \Big\{ X^{\rm G} : X^{\rm G} = \mathbf{diag}\left(X^{\rm G}_{\bf T}, X^{\rm G}_{{\bf S},1}, \ldots, X^{\rm G}_{{\bf S},L}\right), \ X^{\rm G}_{\bf T} \in \mathbb{R}_{\bf S}^{\bar{m}_0^{\rm G} \times \bar{m}_0^{\rm G}}, X^{\rm G}_{\bf T} > 0, X^{\rm G}_{{\bf S},i} \in \mathbb{R}_{\bf S}^{\bar{m}_i^{\rm G} \times \bar{m}_i^{\rm G}} \Big\} \tag{74}$$

$$\mathcal{X}^{\rm K} := \Big\{ X^{\rm K} : X^{\rm K} = \mathbf{diag}\left(X^{\rm K}_{\bf T}, X^{\rm K}_{{\bf S},1}, \ldots, X^{\rm K}_{{\bf S},L}\right), \ X^{\rm K}_{\bf T} \in \mathbb{R}_{\bf S}^{\bar{m}_0^{\rm K} \times \bar{m}_0^{\rm K}}, X^{\rm K}_{\bf T} > 0, X^{\rm K}_{{\bf S},i} \in \mathbb{R}_{\bf S}^{\bar{m}_i^{\rm K} \times \bar{m}_i^{\rm K}} \Big\} \tag{75}$$

$$\mathcal{X}^{\rm GK} := \Big\{ X^{\rm GK} : X^{\rm GK} = \mathbf{diag}\left(X^{\rm GK}_{\bf T}, X^{\rm GK}_{{\bf S},1}, \ldots, X^{\rm GK}_{{\bf S},L}\right), \ X^{\rm GK}_{\bf T} \in \mathbb{R}^{\bar{m}_0^{\rm G} \times \bar{m}_0^{\rm K}}, X^{\rm GK}_{{\bf S},i} \in \mathbb{R}^{\bar{m}_i^{\rm G} \times \bar{m}_i^{\rm K}} \Big\}. \tag{76}$$

We have the following Lemma.

Lemma 2: Let $\bar{m}_0^G, \ldots, \bar{m}_L^G$ be fixed. Given X^G and Y^G in $\mathcal{X}^G$, there exists $\bar{m}_0^K, \ldots, \bar{m}_L^K$, X^K and Y^K in $\mathcal{X}^K$, and X^{GK} and Y^{GK} in $\mathcal{X}^{GK}$ such that

$$\begin{bmatrix} X^G & X^{GK} \\ (X^{GK})^* & X^K \end{bmatrix}^{-1} = \begin{bmatrix} Y^G & Y^{GK} \\ (Y^{GK})^* & Y^K \end{bmatrix} \tag{77}$$

if and only if

$$\begin{bmatrix} X^G_{\bf T} & I \\ I & Y^G_{\bf T} \end{bmatrix} \geq 0. \tag{78}$$

Furthermore, one may choose $\bar{m}_i^K = \mathrm{Rank}(I - Y^G_{{\bf S},i} X^G_{{\bf S},i})$, $\bar{m}_0^K = \mathrm{Rank}(I - Y^G_{\bf T} X^G_{\bf T})$.

Proof: Due to the structure of the matrices, (78) is equivalent to

$$\begin{bmatrix} X^{\rm G}_{\bf T} & X^{\rm GK}_{\bf T} \\ \left(X^{\rm GK}_{\bf T}\right)^* & X^{\rm K}_{\bf T} \end{bmatrix}^{-1} = \begin{bmatrix} Y^{\rm G}_{\bf T} & Y^{\rm GK}_{\bf T} \\ \left(Y^{\rm GK}_{\bf T}\right)^* & Y^{\rm K}_{\bf T} \end{bmatrix} \tag{79}$$

and

$$\begin{bmatrix} X^{\rm G}_{{\bf S},i} & X^{\rm GK}_{{\bf S},i} \\ \left(X^{\rm GK}_{{\bf S},i}\right)^* & X^{\rm K}_{{\bf S},i} \end{bmatrix}^{-1} = \begin{bmatrix} Y^{\rm G}_{{\bf S},i} & Y^{\rm GK}_{{\bf S},i} \\ \left(Y^{\rm GK}_{{\bf S},i}\right)^* & Y^{\rm K}_{{\bf S},i} \end{bmatrix}. \tag{80}$$

The equivalence of the inequality (78) and $\bar{m}_0^{\rm K} = \mathrm{Rank}(I - Y^{\rm G}_{\bf T} X^{\rm G}_{\bf T})$ to the existence of $X^{\rm K}_{\bf T}, X^{\rm GK}_{\bf T}, Y^{\rm K}_{\bf T}$, and $Y^{\rm GK}_{\bf T}$ such that (79) is satisfied is proved in [34] and [24]. To complete the proof, apply the following proposition, whose proof may be found in the Appendix, to $X^{\rm G}_{{\bf S},i}$ and $Y^{\rm G}_{{\bf S},i}$.

Proposition 3: Given R_1, S_1 in $\mathbb{R}_{\bf S}^{n \times n}$, let $k = \mathrm{Rank}(I - R_1 S_1)$. Then, there exist R_2, S_2 in $\mathbb{R}^{n \times k}$ and R_3, S_3 in $\mathbb{R}_{\bf S}^{k \times k}$ such that

$$\begin{bmatrix} S_1 & S_2 \\ S_2^* & S_3 \end{bmatrix}^{-1} = \begin{bmatrix} R_1 & R_2 \\ R_2^* & R_3 \end{bmatrix}. \tag{81}$$

■

We are now in a position to state the main result of this section.

Theorem 3: Let $\bar{\mathcal{M}}^G$ be given. Let the columns of $\mathcal{N}_Y$ form a basis for the null space of $[(\bar{B}^G_{\bf u\bullet})^* \quad (\bar{D}^G_{\bf zu})^*]$, and the columns of $\mathcal{N}_X$ form a basis for the null space of $[\bar{C}^G_{\bf y\bullet} \quad \bar{D}^G_{\bf yd}]$. Then, there exist $\bar{m}_i^K \leq \bar{m}_i^G$, $X^G \in \mathcal{X}^G$, $X^K \in \mathcal{X}^K$, $X^{GK} \in \mathcal{X}^{GK}$, and $\bar{A}^K, \bar{B}^K, \bar{C}^K, \bar{D}^K$ such that the inequality in (67) is satisfied if and only if there exist X^G and Y^G in $\mathcal{X}^G$ such that the three LMIs shown in (82)–(84) at the bottom of the next page are satisfied.

Proof: A direct application of the results in [24] yields the inequalities (82) and (83) and the coupling conditions in (77). We may then invoke Lemma 2 to yield the required result. ■

Remarks:

- If the LMIs of Theorem 3 have feasible solutions $X^{\rm G}$ and $Y^{\rm G}$, one may construct $\bar{A}^{\rm K}, \bar{B}^{\rm K}, \bar{C}^{\rm K}, \bar{D}^{\rm K}$ by first solving for $\bar{X}$ via Lemma 2, and then solving the inequality in (67), which is affine in Θ.

- Note that X^{G} and Y^{G} are coupled only through $X^{\mathrm{G}}_{\mathbf{T}}$ and $Y^{\mathrm{G}}_{\mathbf{T}}$ in the above inequalities. This is intimately tied to the definition of stability and the physical observation that a change of independent variables $s_i \longrightarrow -s_i$ does not in any way change the problem formulation; the problem is spatially symmetric. This same argument cannot be applied to the temporal independent variable t.
- The system of LMIs can be used to construct a controller that stabilizes the system and that results in an induced gain less than one. The performance index, which we have assumed to be one, can be readily changed to γ (i.e., the induced gain is less than γ) by replacing the $-I$ terms in (82) and (83) by $-\gamma I$. The performance level γ can thus become a decision variable which can be minimized.
- Like the analysis LMI, the synthesis condition is valid for both periodic and infinite interconnections, and is independent of the number of blocks in a periodic interconnection. The size of the resulting LMI is only a function of the size of the basic building block used to describe the interconnection. This has an obvious application to reconfigurability: elements can be added or removed without affecting the well-posedness, stability, and performance of the closed-loop interconnection.

We conclude this section with the following proposition, which states that the feasibility of the analysis, and hence synthesis, LMIs is coordinate independent. This fact is used in the next section for controller implementation.

Proposition 4: Let the following data be given.

- $\bar{\mathcal{M}}_1^G = \{\bar{A}_1^G, \bar{B}_1^G, \bar{C}_1^G, \bar{D}_1^G, \bar{\mathbf{m}}_1^G\}, \bar{\mathcal{M}}_1^K = \{\bar{A}_1^K, \bar{B}_1^K, \bar{C}_1^K, \bar{D}_1^K, \bar{\mathbf{m}}_1^K\}$.
- $T^G = \mathbf{diag}(T_0^G, \ldots, T_L^G), T_j^G$ invertible, $T_j^G \in \mathbb{R}_{\mathbf{S}}^{\bar{m}_j^G \times \bar{m}_j^G}$.
- $T^K = \mathbf{diag}(T_0^K, \ldots, T_L^K), T_j^K$ invertible, $T_j^K \in \mathbb{R}_{\mathbf{S}}^{\bar{m}_j^K \times \bar{m}_j^K}$.
- $X = \mathbf{diag}(X_{\mathbf{T}}, X_{\mathbf{S}}), X_{\mathbf{T}} \in \mathbb{X}_{\mathbf{T}}, X_{\mathbf{S}} \in \mathbb{X}_{\mathbf{S}}$.

Define

- $\bar{\mathcal{M}}_2^G = \{\bar{A}_2^G, \bar{B}_2^G, \bar{C}_2^G, \bar{D}_2^G, \bar{\mathbf{m}}_2^G\} := \{(T^G)^{-1}\bar{A}_1^G T^G, (T^G)^{-1}\bar{B}_1^G, \bar{C}_1^G T^G, \bar{D}_1^G, \bar{\mathbf{m}}_1^G\}$.
- $\bar{\mathcal{M}}_2^K = \{\bar{A}_2^K, \bar{B}_2^K, \bar{C}_2^K, \bar{D}_2^K, \bar{\mathbf{m}}_2^K\} := \{(T^K)^{-1}\bar{A}_1^K T^K, (T^K)^{-1}\bar{B}_1^K, \bar{C}_1^K T^K, \bar{D}_1^K, \bar{\mathbf{m}}_1^K\}$.
- $\bar{\mathcal{M}}_1 = \{\bar{A}_1, \bar{B}_1, \bar{C}_1, \bar{D}_1, \bar{\mathbf{m}}_1\} := f_{\mathrm{IC}}(\bar{\mathcal{M}}_1^G, \bar{\mathcal{M}}_1^K)$.
- $\bar{\mathcal{M}}_2 = \{\bar{A}_2, \bar{B}_2, \bar{C}_2, \bar{D}_2, \bar{\mathbf{m}}_2\} := f_{\mathrm{IC}}(\bar{\mathcal{M}}_2^G, \bar{\mathcal{M}}_2^K)$.
- $\tilde{X} = (PTP^*)^* X PTP^*$, where the permutation matrix P is defined in Section 2, and $T = \mathbf{diag}(T^G, T^K)$.

Then

1) $\tilde{X} = \mathbf{diag}(\tilde{X}_{\mathbf{T}}, \tilde{X}_{\mathbf{S}})$, where $\tilde{X}_{\mathbf{T}} \in \mathbb{X}_{\mathbf{T}}, \tilde{X}_{\mathbf{S}} \in \mathbb{X}_{\mathbf{S}}$.

2)
$$\begin{bmatrix} \bar{A}_1^* X + X\bar{A}_1 & X\bar{B}_1 & \bar{C}_1^* \\ \bar{B}_1^* X & -I & \bar{D}_1^* \\ \bar{C}_1 & \bar{D}_1 & -I \end{bmatrix} < 0 \text{ if and only if}$$
$$\begin{bmatrix} \bar{A}_2^* \tilde{X} + \tilde{X}\bar{A}_2 & \tilde{X}\bar{B}_2 & \bar{C}_2^* \\ \bar{B}_2^* \tilde{X} & -I & \bar{D}_2^* \\ \bar{C}_2 & \bar{D}_2 & -I \end{bmatrix} < 0. \tag{85}$$

Proof: Condition 1) follows from direct substitution, while condition 2) follows from the following equality:

$$\begin{bmatrix} \bar{A}_2^* \tilde{X} + \tilde{X}\bar{A}_2 & \tilde{X}\bar{B}_2 & \bar{C}_2^* \\ \bar{B}_2^* \tilde{X} & -I & \bar{D}_2^* \\ \bar{C}_2 & \bar{D}_2 & -I \end{bmatrix} = \begin{bmatrix} PTP^* & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^* \begin{bmatrix} \bar{A}_1^* X + X\bar{A}_1 & X\bar{B}_1 & \bar{C}_1^* \\ \bar{B}_1^* X & -I & \bar{D}_1^* \\ \bar{C}_1 & \bar{D}_1 & -I \end{bmatrix} \times \begin{bmatrix} PTP^* & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$ ■

VI. Controller Implementation

The synthesis procedure of Section V yields $\bar{\mathcal{M}}^{\mathrm{K}}$. There are two issues that need to be resolved.

1) Construct $\mathcal{M}^{\mathrm{K}}$ such that $f_{\mathrm{D2C}}(\mathcal{M}^{\mathrm{K}}) = \bar{\mathcal{M}}^{\mathrm{K}}$.
2) Outline a method for *implementing* the equations associated with the control system $\mathcal{M}^{\mathrm{K}}$.

As we shall see, these two issues are intimately related.

Constructing $\mathcal{M}^{\mathrm{K}}$ essentially consists of finding the inverse of function f_{D2C}. It is easy to see, however, that such an inverse is not unique, since there are many ways in which H can be defined and be consistent with $\bar{\mathbf{m}}^{\mathrm{K}}$ (see Definition 4). More intuitively, in terms of the operators $\mathbf{S}_i$, we have the freedom of expressing the controller equations in terms of $\mathbf{S}_i$ or $\mathbf{S}_i^{-1}$. For example, consider the following two systems of equations, where the signals are only a function of one spatial independent variable s:

$$\begin{aligned} (\mathbf{S}v_+)(s) &= 2v_+(s) + y(s) \\ u(s) &= v_+(s) \\ (\mathbf{S}^{-1}v_-)(s) &= 0.5v_-(s) - 0.5y(s) \\ u(s) &= 0.5v_-(s) - 0.5y(s). \end{aligned} \tag{86}$$

$$\begin{bmatrix} \mathcal{N}_Y & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \begin{bmatrix} \bar{A}^G Y^G + Y^G (\bar{A}^G)^* & Y^G \left(\bar{C}^G_{\mathbf{z}\bullet}\right)^* \\ \bar{C}^G_{\mathbf{z}\bullet} Y^G & -I \end{bmatrix} & \begin{bmatrix} \bar{B}^G_{\bullet\mathbf{d}} \\ \bar{D}^G_{\mathbf{zd}} \end{bmatrix} \\ \begin{bmatrix} \left(\bar{B}^G_{\bullet\mathbf{d}}\right)^* & \left(\bar{D}^G_{\mathbf{zd}}\right)^* \end{bmatrix} & -I \end{bmatrix} \begin{bmatrix} \mathcal{N}_Y & 0 \\ 0 & I \end{bmatrix} < 0 \tag{82}$$

$$\begin{bmatrix} \mathcal{N}_X & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \begin{bmatrix} (\bar{A}^G)^* X^G + X^G \bar{A}^G & X^G \bar{B}^G_{\bullet\mathbf{d}} \\ \left(\bar{B}^G_{\bullet\mathbf{d}}\right)^* X^G & -I \end{bmatrix} & \begin{bmatrix} \left(\bar{C}^G_{\mathbf{z}\bullet}\right)^* \\ \left(\bar{D}^G_{\mathbf{zd}}\right)^* \end{bmatrix} \\ \begin{bmatrix} \bar{C}^G_{\mathbf{z}\bullet} & \bar{D}^G_{\mathbf{zd}} \end{bmatrix} & -I \end{bmatrix} \begin{bmatrix} \mathcal{N}_X & 0 \\ 0 & I \end{bmatrix} < 0 \tag{83}$$

$$\begin{bmatrix} X^G_{\mathbf{T}} & I \\ I & Y^G_{\mathbf{T}} \end{bmatrix} \geq 0. \tag{84}$$

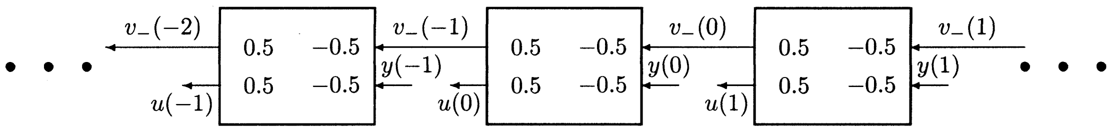

Fig. 10. Signal flow interpretation.

These two systems of equations yield the same input-output behavior from y to u: for every $y \in \ell_2$, there exists a $u \in \ell_2$ that satisfies both system of equations. In fact, the internal variables v_+ and v_- are related by $v_- = \mathbf{S}v_+$. The advantage of the realization on the right-hand side of (86) is that it lends itself to practical implementation, since it can be given a stable, recursive signal flow interpretation. This is depicted in Fig. 10. At each location s signal $u(s)$ is the following quantity:

$$u(s) = -0.5(y(s) + 0.5y(s+1) + 0.25y(s+2) + \cdots + (0.5)^k y(s+k) + \cdots). \quad (87)$$

We do not, however, need to physically connect an infinite number of signals y to our local controller at location s in order to generate $u(s)$. The recursive law on the right-hand side of (86), depicted in Fig. 10, allows us to implement this relation via nearest neighbor coupling with coupling signal $v_-(s)$.

In general, assume that a realization $\mathcal{M}^{\mathrm{K}} = \{A^{\mathrm{K}}, B^{\mathrm{K}}, C^{\mathrm{K}}, D^{\mathrm{K}}, \mathbf{m}^{\mathrm{K}}\}$ has been constructed such that $f_{\mathrm{D2C}}(\mathcal{M}^{\mathrm{K}}) = \bar{\mathcal{M}}^{\mathrm{K}}$. The control system is thus captured by the following equations:

$$\begin{bmatrix} \dot{x}^{\mathrm{K}}(t,\mathbf{s}) \\ (\mathbf{\Delta}_{\mathbf{m}^{\mathrm{K}}} v^{\mathrm{K}}(t))(\mathbf{s}) \\ u(t,\mathbf{s}) \end{bmatrix} = \begin{bmatrix} A^{\mathrm{K}}_{\mathbf{TT}} & A^{\mathrm{K}}_{\mathbf{TS}} & B^{\mathrm{K}}_{\mathbf{T}} \\ A^{\mathrm{K}}_{\mathbf{ST}} & A^{\mathrm{K}}_{\mathbf{SS}} & B^{\mathrm{K}}_{\mathbf{S}} \\ C^{\mathrm{K}}_{\mathbf{T}} & C^{\mathrm{K}}_{\mathbf{S}} & D^{\mathrm{K}} \end{bmatrix} \begin{bmatrix} x^{\mathrm{K}}(t,\mathbf{s}) \\ v^{\mathrm{K}}(t,\mathbf{s}) \\ y(t,\mathbf{s}) \end{bmatrix}. \quad (88)$$

If $(\mathbf{\Delta}_{\mathbf{S},\mathbf{m}^{\mathrm{K}}} - A^{\mathrm{K}}_{\mathbf{SS}})$ is invertible on ℓ_2 and can be expressed as

$$\left(\mathbf{\Delta}_{\mathbf{S},\mathbf{m}^{\mathrm{K}}} - A^{\mathrm{K}}_{\mathbf{SS}}\right)^{-1} = \left(\sum_{k=0}^{\infty} \left((\mathbf{\Delta}_{\mathbf{S},\mathbf{m}^{\mathrm{K}}})^{-1} A^{\mathrm{K}}_{\mathbf{SS}}\right)^k\right) (\mathbf{\Delta}_{\mathbf{S},\mathbf{m}^{\mathrm{K}}})^{-1} \quad (89)$$

the control system can be implemented by interconnecting the following finite-dimensional, linear time-invariant subsystems:

$$\begin{bmatrix} \dot{x}^{\mathrm{K}}(t,\mathbf{s}) \\ w^{\mathrm{K}}(t,\mathbf{s}) \\ u(t,\mathbf{s}) \end{bmatrix} = \begin{bmatrix} A^{\mathrm{K}}_{\mathbf{TT}} & A^{\mathrm{K}}_{\mathbf{TS}} & B^{\mathrm{K}}_{\mathbf{T}} \\ A^{\mathrm{K}}_{\mathbf{ST}} & A^{\mathrm{K}}_{\mathbf{SS}} & B^{\mathrm{K}}_{\mathbf{S}} \\ C^{\mathrm{K}}_{\mathbf{T}} & C^{\mathrm{K}}_{\mathbf{S}} & D^{\mathrm{K}} \end{bmatrix} \begin{bmatrix} x^{\mathrm{K}}(t,\mathbf{s}) \\ v^{\mathrm{K}}(t,\mathbf{s}) \\ y(t,\mathbf{s}) \end{bmatrix} \quad (90)$$

with the nearest neighbor coupling law $(\mathbf{\Delta}_{\mathbf{S},\mathbf{m}} v^{\mathrm{K}}(t))(\mathbf{s}) = w^{\mathrm{K}}(t,\mathbf{s})$. It should be stressed that even though the controllers are coupled only to their nearest neighbors, the *overall* transfer of information can occur over larger distances.

The controller implementation problem thus consists of constructing $\mathcal{M}^{\mathrm{K}} = \{A^{\mathrm{K}}, B^{\mathrm{K}}, C^{\mathrm{K}}, D^{\mathrm{K}}, \mathbf{m}^{\mathrm{K}}\}$ such that $f_{\mathrm{D2C}}(\mathcal{M}^{\mathrm{K}}) = \bar{\mathcal{M}}^{\mathrm{K}}$ *and* such that the operator $(\mathbf{\Delta}_{\mathbf{S},\mathbf{m}^{\mathrm{K}}} - A^{\mathrm{K}}_{\mathbf{SS}})^{-1}$ can be expanded as per (89). The following procedure will achieve this task.

Step 1) Given $\bar{\mathcal{M}}^{\mathrm{K}} = \{\bar{A}^{\mathrm{K}}, \bar{B}^{\mathrm{K}}, \bar{C}^{\mathrm{K}}, \bar{D}^{\mathrm{K}}, \bar{\mathbf{m}}^{\mathrm{K}}\}$, define the following set of scaling matrices:

$$\mathcal{X}^{\mathrm{K}}_{\mathbf{S}} := \left\{X^{\mathrm{K}}_{\mathbf{S}} : X^{\mathrm{K}}_{\mathbf{S}} = \mathbf{diag}\left(X^{\mathrm{K}}_{\mathbf{S},1}, \ldots, X^{\mathrm{K}}_{\mathbf{S},L}\right), X^{\mathrm{K}}_{\mathbf{S},i} \in \mathbb{R}_{\mathbf{S}}^{\bar{m}^{\mathrm{K}}_i \times \bar{m}^{\mathrm{K}}_i}\right\}. \quad (91)$$

Solve the following LMI for $X^{\mathrm{K}}_{\mathbf{S}} \in \mathcal{X}^{\mathrm{K}}_{\mathbf{S}}$:

$$\left(\bar{A}^{\mathrm{K}}_{\mathbf{SS}}\right)^* X^{\mathrm{K}}_{\mathbf{S}} + X^{\mathrm{K}}_{\mathbf{S}} \bar{A}^{\mathrm{K}}_{\mathbf{SS}} < 0 \quad (92)$$

where without loss of generality, $X^{\mathrm{K}}_{\mathbf{S}}$ is invertible. This LMI may not have a solution; see the remarks at the end of this procedure for a detailed discussion of when the LMI is guaranteed to have a solution.

Step 2) Factor each $X^{\mathrm{K}}_{\mathbf{S},i}$ as follows:

$$X^{\mathrm{K}}_{\mathbf{S},i} = \left(T^{\mathrm{K}}_i\right)^* \begin{bmatrix} I_{m^{\mathrm{K}}_i} & 0 \\ 0 & -I_{m^{\mathrm{K}}_{-i}} \end{bmatrix} T^{\mathrm{K}}_i \quad (93)$$

where $m^{\mathrm{K}}_i + m^{\mathrm{K}}_{-i} = \bar{m}^{\mathrm{K}}_i$. Define $T^{\mathrm{K}}_0 := I_{\bar{m}_0}, T^{\mathrm{K}} := \mathbf{diag}(T^{\mathrm{K}}_0, \ldots, T^{\mathrm{K}}_L)$, and apply the following coordinate transformation to $\bar{\mathcal{M}}^{\mathrm{K}}$:

$$\bar{\mathcal{M}}^{\mathrm{K}} = \{\bar{A}^{\mathrm{K}}, \bar{B}^{\mathrm{K}}, \bar{C}^{\mathrm{K}}, \bar{D}^{\mathrm{K}}, \bar{\mathbf{m}}^{\mathrm{K}}\} \mapsto \{(T^{\mathrm{K}})^{-1}\bar{A}^{\mathrm{K}}T^{\mathrm{K}}, (T^{\mathrm{K}})^{-1}\bar{B}^{\mathrm{K}}, \bar{C}^{\mathrm{K}}T^{\mathrm{K}}, \bar{D}^{\mathrm{K}}, \bar{\mathbf{m}}^{\mathrm{K}}\}. \quad (94)$$

By Proposition 4, this change of coordinates still yields a suitable realization $\bar{\mathcal{M}}^{\mathrm{K}}$. Note that in this coordinate system

$$\left(\bar{A}^{\mathrm{K}}_{\mathbf{SS}}\right)^* H^{\mathrm{K}} + H^{\mathrm{K}} \bar{A}^{\mathrm{K}}_{\mathbf{SS}} < 0 \quad (95)$$

where H^{K} is defined analogously to H in (44).

Step 3) Define $m^{\mathrm{K}}_0 := \bar{m}^{\mathrm{K}}_0$ and $\mathbf{m}^{\mathrm{K}} := (m^{\mathrm{K}}_0, m^{\mathrm{K}}_1, m^{\mathrm{K}}_{-1}, \ldots, m^{\mathrm{K}}_{-L})$, where the m^{K}_i and m^{K}_{-i} were defined implicitly in (93). Note that if the inequality in (95) is satisfied, $(\bar{A}^{\mathrm{K}}_{\mathbf{SS}} H^K - I)$ is invertible; we can thus solve for $A^{\mathrm{K}}, B^{\mathrm{K}}, C^{\mathrm{K}}$, and D^{K} in Definition 4 in terms of $\bar{A}^{\mathrm{K}}, \bar{B}^{\mathrm{K}}, \bar{C}^{\mathrm{K}}, \bar{D}^{\mathrm{K}}$, and H^{K} to yield

$$A^{\mathrm{K}}_{\mathbf{SS}} := \left(\bar{A}^{\mathrm{K}}_{\mathbf{SS}} H^{\mathrm{K}} - I\right)^{-1} \left(\bar{A}^{\mathrm{K}}_{\mathbf{SS}} H^{\mathrm{K}} + I\right) \quad (96)$$

$$\begin{bmatrix} A^{\mathrm{K}}_{\mathbf{ST}} & B^{\mathrm{K}}_{\mathbf{S}} \end{bmatrix} := -\sqrt{2}\left(\bar{A}^{\mathrm{K}}_{\mathbf{SS}} H^{\mathrm{K}} - I\right)^{-1} \begin{bmatrix} \bar{A}^{\mathrm{K}}_{\mathbf{ST}} & \bar{B}^{\mathrm{K}}_{\mathbf{S}} \end{bmatrix} \quad (97)$$

$$\begin{bmatrix} A^{\mathrm{K}}_{\mathbf{TS}} \\ C^{\mathrm{K}}_{\mathbf{S}} \end{bmatrix} := \sqrt{2} \begin{bmatrix} \bar{A}^{\mathrm{K}}_{\mathbf{TS}} \\ \bar{C}^{\mathrm{K}}_{\mathbf{S}} \end{bmatrix} H^{\mathrm{K}} \left(\bar{A}^{\mathrm{K}}_{\mathbf{SS}} H^{\mathrm{K}} - I\right)^{-1} \quad (98)$$

$$\begin{bmatrix} A^{\mathrm{K}}_{\mathbf{TT}} & B^{\mathrm{K}}_{\mathbf{T}} \\ C^{\mathrm{K}}_{\mathbf{T}} & D^{\mathrm{K}} \end{bmatrix} := \begin{bmatrix} \bar{A}^{\mathrm{K}}_{\mathbf{TT}} & \bar{B}^{\mathrm{K}}_{\mathbf{T}} \\ \bar{C}^{\mathrm{K}}_{\mathbf{T}} & \bar{D}^{\mathrm{K}} \end{bmatrix} - \begin{bmatrix} \bar{A}^{\mathrm{K}}_{\mathbf{TS}} \\ \bar{C}^{\mathrm{K}}_{\mathbf{S}} \end{bmatrix} \times H^{\mathrm{K}} \left(\bar{A}^{\mathrm{K}}_{\mathbf{SS}} H^{\mathrm{K}} - I\right)^{-1} \begin{bmatrix} \bar{A}^{\mathrm{K}}_{\mathbf{ST}} & \bar{B}^{\mathrm{K}}_{\mathbf{S}} \end{bmatrix} \quad (99)$$

Define $\mathcal{M}^{\mathrm{K}} := \{A^{\mathrm{K}}, B^{\mathrm{K}}, C^{\mathrm{K}}, D^{\mathrm{K}}, \mathbf{m}^{\mathrm{K}}\}$; by construction, $f_{\mathrm{D2C}}(\mathcal{M}^{\mathrm{K}}) = \bar{\mathcal{M}}^{\mathrm{K}}$. Also, note that by construction, $(I - A^{\mathrm{K}}_{\mathbf{SS}})$ is invertible, since $I - A^{\mathrm{K}}_{\mathbf{SS}} = -2(\bar{A}^{\mathrm{K}}_{\mathbf{SS}} H^{\mathrm{K}} - I)^{-1}$, and thus by Proposition 2, Theorem 2, and Theorem 1, the control system $\mathcal{M}^{\mathrm{K}}$ solves the problem formulation of Section V. In addition, note that

$$A^{\mathrm{K}}_{\mathbf{SS}}\left(A^{\mathrm{K}}_{\mathbf{SS}}\right)^* - I = 2\left(\bar{A}^{\mathrm{K}}_{\mathbf{SS}} H^{\mathrm{K}} - I\right)^{-1} H^{\mathrm{K}} \times \left(\left(\bar{A}^{\mathrm{K}}_{\mathbf{SS}}\right)^* H^{\mathrm{K}} + H^{\mathrm{K}} \bar{A}^{\mathrm{K}}_{\mathbf{SS}}\right) H^{\mathrm{K}} \left(\bar{A}^{\mathrm{K}}_{\mathbf{SS}} H^{\mathrm{K}} - I\right)^{-*} \quad (100)$$

which by (95) implies that $\bar{\sigma}(A^{\mathrm{K}}_{\mathbf{SS}}) < 1$. Finally, since $(\boldsymbol{\Delta}_{\mathbf{S},\mathbf{m}^{\mathrm{K}}})^{-1}$ is a bounded, unitary operator and

$$\boldsymbol{\Delta}_{\mathbf{S},\mathbf{m}^{\mathrm{K}}} - A^{\mathrm{K}}_{\mathbf{SS}} = \boldsymbol{\Delta}_{\mathbf{S},\mathbf{m}^{\mathrm{K}}}\left(I - (\boldsymbol{\Delta}_{\mathbf{S},\mathbf{m}^{\mathrm{K}}})^{-1}A^{\mathrm{K}}_{\mathbf{SS}}\right) \quad (101)$$

operator $(\boldsymbol{\Delta}_{\mathbf{S},\mathbf{m}^{\mathrm{K}}} - A^{\mathrm{K}}_{\mathbf{SS}})^{-1}$ exists, and can be expressed as in (89) (see [6, p. 169], for example), as required.

Remarks:

- The above procedure can only be performed if the well-posedness LMI in Step 1) has a solution. This is not guaranteed, even if $\bar{A}^{\mathrm{K}}_{\mathbf{SS}}$ is perturbed by a small amount, since the scaling matrix X^{K} is structured. There are some instances, however, when such a scaling matrix X^{K} is guaranteed to exist.
  1) If there is only one spatial dimension, a scale exists if and only if $\bar{A}^{\mathrm{K}}_{\mathbf{SS}}$ has no purely imaginary or zero eigenvalues, which can always be ensured by a small perturbation of $\bar{A}^{\mathrm{K}}_{\mathbf{SS}}$.
  2) If matrix $\bar{A}^{\mathrm{K}}_{\mathbf{SS}}$ is block diagonal, the factorization problem reduces to L independent LMIs, which always have a solution as argued beforehand. The second example in Section VII results in a block diagonal $\bar{A}^{\mathrm{K}}_{\mathbf{SS}}$, for example.
  3) If the open-loop plant data satisfies

$$C^{\mathrm{G}}_{\mathbf{S},\mathbf{y}} = 0 \quad D^{\mathrm{G}}_{\mathbf{yu}} = 0 \quad (102)$$

it can readily be shown that the resulting closed-loop system $A_{\mathbf{SS}}$ matrix is

$$A_{\mathbf{SS}} = \begin{bmatrix} P^{\mathrm{G}}_{\mathbf{S}} & P^{\mathrm{K}}_{\mathbf{S}} \end{bmatrix} \begin{bmatrix} A^{\mathrm{G}}_{\mathbf{SS}} & B^{\mathrm{G}}_{\mathbf{S},\mathbf{u}}C^{\mathrm{K}}_{\mathbf{S}} \\ 0 & A^{\mathrm{K}}_{\mathbf{SS}} \end{bmatrix} \begin{bmatrix} P^{\mathrm{G}}_{\mathbf{S}} & P^{\mathrm{K}}_{\mathbf{S}} \end{bmatrix}^* \quad (103)$$

where $P^{\mathrm{G}}_{\mathbf{S}}$ and $P^{\mathrm{K}}_{\mathbf{S}}$ are the permutation matrices defined in (57). It is then easy to show that if the closed-loop system matrix $A_{\mathbf{SS}}$ satisfies the well-posedness LMI (43), then so must $A^{\mathrm{G}}_{\mathbf{SS}}$ and $A^{\mathrm{K}}_{\mathbf{SS}}$. Since the synthesis LMIs guarantee that the well-posedness LMI is satisfied for the closed-loop system, it then follows that the well-posedness LMI for the controller in Step 1) will have a solution.

The physical interpretation of the $C^{\mathrm{G}}_{\mathbf{S},\mathbf{y}} = 0$ assumption is that nearest neighbor information cannot directly affect the sensor signals, but must rather go through some temporal dynamics first. Similarly, the physical interpretation of the $D^{\mathrm{G}}_{\mathbf{yu}} = 0$ assumption is that the actuator signals cannot directly affect the sensor signals.

A dual result holds for the assumption

$$B^{\mathrm{G}}_{\mathbf{S},\mathbf{u}} = 0 \quad D^{\mathrm{G}}_{\mathbf{yu}} = 0 \quad (104)$$

where the resulting closed-loop system $A_{\mathbf{SS}}$ matrix is

$$A_{\mathbf{SS}} = \begin{bmatrix} P^{\mathrm{G}}_{\mathbf{S}} & P^{\mathrm{K}}_{\mathbf{S}} \end{bmatrix} \begin{bmatrix} A^{\mathrm{G}}_{\mathbf{SS}} & 0 \\ B^{\mathrm{K}}_{\mathbf{S},\mathbf{u}}C^{\mathrm{G}}_{\mathbf{S}} & A^{\mathrm{K}}_{\mathbf{SS}} \end{bmatrix} \begin{bmatrix} P^{\mathrm{G}}_{\mathbf{S}} & P^{\mathrm{K}}_{\mathbf{S}} \end{bmatrix}^*. \quad (105)$$

- The nearest neighbor information transfer captured by $(\boldsymbol{\Delta}_{\mathbf{S},\mathbf{m}}v^{\mathrm{K}}(t))(\mathbf{s}) = w^{\mathrm{K}}(t,\mathbf{s})$ is assumed to be instantaneous; in practice, there will be delays and distortion in the transfer of information, due to the nonidealities of any real communication channel. A method for analyzing systems with these nonidealities may be found in [35], and a discrete time algorithm for transferring nearest neighbor information is presented in [11].

VII. Examples

We applied the techniques developed in this paper to two problems. The first consists of a numerical problem in two spatial dimensions. The second consists of control design for a finite difference approximation of the two-dimensional heat equation; this example was chosen to suggest how the tools and techniques in this paper could be extended to control systems governed by PDEs.

A. Numerical Example in Two Spatial Dimensions

Consider the following system equations, expressed in operator form for brevity:

$$\begin{aligned} \ddot{p} &= \frac{1}{8}\left(\mathbf{S}_1 + \mathbf{S}_1^{-1} + \mathbf{S}_2 + \mathbf{S}_2^{-1} + 4\right)p \\ &\quad + \frac{1}{16}\left(\mathbf{S}_1 + \mathbf{S}_1^{-1} - 2\right)\left(\mathbf{S}_2 + \mathbf{S}_2^{-1} - 2\right)d_1 + u \\ z_1 &= \frac{1}{16}\left(\mathbf{S}_1 + \mathbf{S}_1^{-1} - 2\right)\left(\mathbf{S}_2 + \mathbf{S}_2^{-1} - 2\right)p \\ z_2 &= u \\ y &= p + d_2. \end{aligned}$$

Each signal is a function of one temporal independent variable, and two spatial independent variables: $p = p(t, s_1, s_2)$, etc. A realization of this system, as per (52), can readily be constructed using the software package described in [13]. It has two temporal states $x(t, s_1, s_2)$ ($A^{\mathrm{G}}_{\mathbf{TT}}$ is a two by two matrix), and each of the interconnection variables $v_{+,1}(t, s_1, s_2), v_{-,1}(t, s_1, s_2), v_{+,2}(t, s_1, s_2)$, and $v_{-,2}(t, s_1, s_2)$ is of size two ($A^{\mathrm{G}}_{\mathbf{SS}}$ is an eight by eight matrix); the details are omitted.

Some things to note about the example are as follows.

1) The disturbance d_1 acts through a spatial high-pass filter. In particular, the filter completely rejects disturbances that are constant in space, but passes through disturbances whose entries alternate in sign with their nearest neighbors. For example, focusing in on a three by three grid, this high frequency disturbance would have the following profile:

$$d_1(t) = \begin{bmatrix} & & \vdots & & \\ & f(t) & -f(t) & f(t) & \\ \cdots & -f(t) & f(t) & -f(t) & \cdots \\ & f(t) & -f(t) & f(t) & \\ & & \vdots & & \end{bmatrix} \quad (106)$$

where $f(t)$ is some function of time.

2) The same spatial filter is used to define error variable z_1. We are thus interested in rejecting high spatial frequency variations of variable p.

3) The second error variable z_2 is the control effort u. The sensor signal y is simply p corrupted by noise d_2. The control signal u acts directly on the $\ddot{p}$ equation.

4) The unforced dynamics

$$\ddot{p} = \frac{1}{8}\left(\mathbf{S}_1 + \mathbf{S}_1^{-1} + \mathbf{S}_2 + \mathbf{S}_2^{-1} + 4\right) p \tag{107}$$

have a simple interpretation: the force on a mass particle at location (s_1, s_2), in a direction orthogonal to the grid, is a function of the difference between the displacements of the particle and its nearest neighbors, in a direction orthogonal to the grid, and is repulsive in nature.

This example is perhaps the simplest, nontrivial applications of the tools presented in this paper. In particular, note the following.

1) It is in two spatial dimensions. An explicit state-space representation of a10×10 grid, for example, would result in a 200 by 200 state transition matrix.
2) Spatial filters are used to shape the input disturbance, and define the performance objective.

The price to be paid for this simplicity, however, is physical relevance. While one could readily ascribe a physical interpretation to the above equations (a lumped approximation of a membrane under compression, or electro-static forces acting on a two-dimensional array of charged particles), it would not be a realistic one. The reader is referred to [12], [27], and [22] for applications and more realistic examples tackled using these tools.

1) Distributed Control Design: A distributed controller was designed using the control synthesis software described in [13]. The resulting controller had one temporal state $x^{\mathrm{K}}(t, s_1, s_2)$, and each of the interconnection variables $v^{\mathrm{K}}_{+,1}(t, s_1, s_2), v^{\mathrm{K}}_{-,1}(t, s_1, s_2), v^{\mathrm{K}}_{+,2}(t, s_1, s_2)$, and $v^{\mathrm{K}}_{-,2}(t, s_1, s_2)$ was of size two.

It took 0.6 s to design the controller on a Pentium III, 1.13 GHz micro-processor. The upper bound to the $\mathcal{L}_2$ induced gain of the closed-loop system, as provided by the controller synthesis routine, was 4.58. The $\mathcal{L}_2$ induced gain of the system was then calculated to be 4.20 using a frequency search (note that these figures do not have to match, since the analysis LMI in Section IV-D is a sufficient, but not necessary, condition).

2) Decentralized Control Design: Various decentralized controllers were designed by making various simplifications.

Decentralized Controller Number 1: A fully decentralized controller was then extracted from the distributed controller by discarding all interconnection variables. The resulting closed-loop system was unstable.

Decentralized Controller Number 2: A fully decentralized controller was then designed by simplifying the system equations as follows:

$$\begin{aligned} \ddot{p} &= p - d_1 + u \\ z_1 &= -p \\ z_2 &= u \\ y &= p + d_2. \end{aligned}$$

The simplification is obtained by considering the worst case effects of the spatial operators. In terms of the unforced dynamics, the most instability is obtained when all the neighbors are acting in unison. In terms of the disturbance and error variable, the worst case effects occur when neighbors alternate in sign.

The resulting controller was then interconnected with the open-loop plant, and a frequency search used to determine the $\mathcal{L}_2$ gain. The result was 5.74.

Other Decentralized Control Designs: Other decentralized controllers were designed by considering various simplifications of the system equations. They either resulted in an unstable closed-loop system, or in a closed-loop system with a larger $\mathcal{L}_2$ induced gain than that obtained with decentralized controller number 2.

3) Centralized Control Design: Centralized controllers were designed for periodic interconnections of various size (corresponding to the torus in Fig. 4) using the LMI toolbox [25]. The largest size problem that could be solved in a reasonable time was a 3×3 grid, which took 378 s. The resulting $\mathcal{L}_2$ induced gain was 4.02. The controller was a nine-state, nine-input, and nine-output system.

The computation time for a 2×2 grid was 4.14 s, and for a 6×1 grid, it was 44.95 s. By assuming a polynomial growth in computation time as a function of the size of the problem [7], it would take on the order of 5 years to design a centralized controller for a 10×10 grid (this does not take into account computer memory limitations). It should be noted, however, that for periodic interconnections, the spatially invariant structure allows one to use the transform methods in [3], and the computations would be significantly simpler.

4) Summary: For this particular example, the distributed controller resulted in a closed-loop gain which was 1.37 times smaller than that obtained with the best decentralized controller, and 1.05 times larger than that obtained with a centralized controller for a three by three grid.

B. Finite Difference Approximation of a Heat Equation

Consider the following equation which captures the time evolution of the temperature of a bi-infinite dimensional plate:

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial l_1^2} + \frac{\partial^2 U}{\partial l_2^2} + Q \tag{108}$$

where l_1 and l_2 are the spatial independent variables, t is the temporal independent variable, $U(t, l_1, l_2)$ is the temperature of the plate, and $Q(t, l_1, l_2)$ is a distributed heat source. The boundary conditions are taken to be simply $U(t, -\infty, l_2) = U(t, \infty, l_2) = U(t, l_1, -\infty) = U(t, l_1, \infty) = 0$. A central, finite difference approximation of the two spatial partial derivatives results in the following continuous-time, discrete-space approximation:

$$\begin{aligned} \frac{\partial U}{\partial t}(t, s_1, s_2) = {} & U(t, s_1 + 1, s_2) + U(t, s_1 - 1, s_2) \\ & + U(t, s_1, s_2 + 1) + U(t, s_1, s_s - 1) \\ & - 4U(t, s_1, s_2) + Q(t, s_1, s_2) \end{aligned} \tag{109}$$

where the continuous independent variables l_1 and l_2 have been replaced by the discrete independent variables s_1 and s_2, assumed here to take integer values. This may be expressed as

$$\dot{U} = \left(\mathbf{S}_1 + \mathbf{S}_1^{-1} + \mathbf{S}_2 + \mathbf{S}_2^{-1} - 4\right) U + Q. \tag{110}$$

Let d_1 be a heat disturbance, d_2 sensor noise, u the control, z_1 the penalty on the temperature of the plate, z_2 the penalty on the control effort, and y the sensed information:

$$\dot{U} = \left(\mathbf{S}_1 + \mathbf{S}_1^{-1} + \mathbf{S}_2 + \mathbf{S}_2^{-1} - 4\right) U + d_1 + u \quad (111)$$

$$z_1 = 0.1U \quad (112)$$

$$z_2 = u \quad (113)$$

$$y = U + d_2. \quad (114)$$

A distributed controller was designed using the control synthesis software described in [13]. The upper bound to the $\mathcal{L}_2$ gain was found to be $= 1.03$. The lower bound on the achievable performance obtained via a frequency search was 1.02, or less than one percent from the upper bound of 1.03. A realization $\mathcal{M}^{\mathrm{K}}$ was then constructed from $\bar{\mathcal{M}}^{\mathrm{K}}$ as discussed in Section 6. The result was the realization shown in (115)–(117) at the bottom of the page.

The controller equations are structured enough that we may readily express $\mathcal{M}^{\mathrm{K}}$ in input–output form

$$u = -k_1 \left(\frac{d}{dt} - k_2 \left(\frac{1}{\mathbf{S}_1 + k_3} + \frac{1}{\mathbf{S}_{-1} + k_3} + \frac{1}{\mathbf{S}_2 + k_3} + \frac{1}{\mathbf{S}_{-2} + k_3} \right) + k_4 \right)^{-1} y \quad (118)$$

where $k_1 = 1.3153, k_2 = 2.7094, k_3 = 0.4767$, and $k_4 = 10.1598$.

VIII. Concluding Remarks

The results presented in this paper have many natural extensions and applications. A method for incorporating physically motivated boundary conditions, such as Dirichlet and Neumann boundary conditions, is presented in [28]. Discrete time extensions are discussed in [16], [15]. Model reduction is addressed in [5]. The application of these tools to airplanes flying in formation is presented in [23] and [22]. Relaxation of spatial and temporal invariance is addressed in [21], [19], and [20]. Extensions to parameter varying systems is discussed in [43]. An in-depth description of how this framework can be extended to encompass uncertainty may be found in [17]. Other natural extensions, such as nonlinear interconnected systems, are also discussed in [17].

Appendix

Proof of Proposition 1: One direction of the proof is straight-forward. Assume that $(\mathbf{\Delta}_{\mathbf{S},\mathbf{m}} - A_{\mathbf{SS}})$ is invertible on ℓ_2. We can then immediately write down the solution to the system described in Definition 3 (see Section IV-B)

$$x(t) = \int_0^t \exp(\mathbf{A}(t-\tau))\mathbf{B}(d(\tau), n(\tau))\, d\tau \quad (119)$$

$$(w(t), v(t), z(t)) = \mathbf{C}x(t) + \mathbf{D}(d(t), n(t)) \quad (120)$$

where

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} A_{\mathbf{TT}} & [B_{\mathbf{T}} \quad 0] \\ \begin{bmatrix} A_{\mathbf{ST}} \\ 0 \\ C_{\mathbf{T}} \end{bmatrix} & \begin{bmatrix} B_{\mathbf{S}} & 0 \\ 0 & 0 \\ D & 0 \end{bmatrix} \end{bmatrix} + \begin{bmatrix} A_{\mathbf{TS}} \\ \begin{bmatrix} A_{\mathbf{SS}} \\ I \\ C_{\mathbf{S}} \end{bmatrix} \end{bmatrix} \times (\mathbf{\Delta}_{\mathbf{S},\mathbf{m}} - A_{\mathbf{SS}})^{-1} [A_{\mathbf{ST}} \quad [B_{\mathbf{S}}. \quad I]] \quad (121)$$

The result then follows since $\mathbf{A}, \mathbf{B}, \mathbf{C}$, and $\mathbf{D}$ are bounded operators, and $\exp(\mathbf{A}t)$ is bounded on the compact interval $[0, T]$.

Now assume that the interconnection is well-posed, and let $\mathbf{\Theta} = (\mathbf{\Delta}_{\mathbf{S},\mathbf{m}} - A_{\mathbf{SS}})$. For any given p in $\ell_2, \|p\|_{\ell_2} \leq 1$, let $n(t) = \delta p$ for all $t \in [0, T]$. Set $d(t) = 0$ for all $t \in [0, T]$. Since $x(0) = 0, \mathbf{\Theta} v(0) = \delta p$; in addition, $\|v(0)\|_{\ell_2} \leq 1$. By uniqueness, there is only one $v(0)$ that satisfies this equation. To summarize, for all $p \in \ell_2, \|p\|_{\ell_2} \leq 1$, there exists a unique $r \in \ell_2, \|r\|_{\ell_2} \leq 1/\delta$, such that $\mathbf{\Theta} r = p$. By linearity of $\mathbf{\Theta}$, this implies that $\mathbf{\Theta}$ is invertible on ℓ_2, as required. ■

Proof of Theorem 1: We will prove the result in three steps.

1) Show that the system is well-posed; we will do this by explicitly constructing $(\mathbf{\Delta}_{\mathbf{S},\mathbf{m}} - A_{\mathbf{SS}})^{-1}$.
2) Once it has been shown that the system is well-posed, we will show that $\exp(\mathbf{A}t)$ is exponentially stable.
3) Once it has been shown that the system is well-posed and exponentially stable, we may express the system equations as per (18) and (19), where all signals are in $\mathcal{L}_2$; we will then show that $\|z\|_{\mathcal{L}_2}^2 \leq (1-\beta)\|d\|_{\mathcal{L}_2}^2$ for all $d \in \mathcal{L}_2$, where β is some strictly positive constant.

$$\mathbf{m}^{\mathrm{K}} = (1, 1, 1, 1, 1) \quad (115)$$

$$A^{\mathrm{K}} = \begin{bmatrix} -10.1598 & 4.1101 & 4.1101 & 4.1101 & 4.1101 \\ 0.6592 & -0.4767 & 0.0000 & 0.0000 & 0.0000 \\ 0.6592 & 0.0000 & -0.4767 & 0.0000 & 0.0000 \\ 0.6592 & 0.0000 & 0.0000 & -0.4767 & 0.0000 \\ 0.6592 & 0.0000 & 0.0000 & 0.0000 & -0.4767 \end{bmatrix}$$

$$B^{\mathrm{K}} = \begin{bmatrix} 2.8700 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.0000 \end{bmatrix} \quad (116)$$

$$C^{\mathrm{K}} = [-0.4583 \quad 0.0000 \quad 0.0000 \quad 0.0000 \quad 0.0000]$$

$$D^{\mathrm{K}} = 0.0000. \quad (117)$$

Without loss of generality, assume that $X_{\mathbf{S}}$ is invertible; if $X_{\mathbf{S}}$ is not invertible, it can always be perturbed to be made invertible—by adding ϵI, for example— and still result in $J < 0$.

Well-Posedness: We will show this via two propositions. Define $\tilde{\mathbf{\Delta}}$ as follows:

$$\tilde{\mathbf{\Delta}} := \mathbf{diag}\left(\mathbf{S}_1^{-1} I_{m_1+m_{-1}}, \ldots, \mathbf{S}_L^{-1} I_{m_L+m_{-L}}\right). \tag{122}$$

Proposition 5: If $J < 0$, then $(A_{\mathbf{SS}}^{-} - \tilde{\mathbf{\Delta}} A_{\mathbf{SS}}^{+})$ is invertible on ℓ_2.

Proof of Proposition 5: Define

$$N := \left(A_{\mathbf{SS}}^{+}\right)^{*} X_{\mathbf{S}} A_{\mathbf{SS}}^{+} - \left(A_{\mathbf{SS}}^{-}\right)^{*} X_{\mathbf{S}} A_{\mathbf{SS}}^{-}. \tag{123}$$

The (2, 2) block of matrix J is simply $N + C_{\mathbf{S}}^{*} C_{\mathbf{S}}$; it thus follows that $N < 0$ if $J < 0$. Matrix $X_{\mathbf{S}}$ can be factored as $X_{\mathbf{S}} = T^{*} Q^{*} R Q T$, where $R = \mathbf{diag}(I, -I), T$ is invertible and commutes with $\tilde{\mathbf{\Delta}}$, and Q is a permutation matrix which reorders the columns of R. Define

$$\hat{A} = \begin{bmatrix} \hat{A}_1 \\ \hat{A}_2 \end{bmatrix} := QTA_{\mathbf{SS}}^{+}(QT)^{-1}$$
$$\hat{E} = \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \end{bmatrix} := QTA_{\mathbf{SS}}^{-}(QT)^{-1}. \tag{124}$$

The condition $N < 0$ is thus equivalent to

$$\begin{bmatrix} \hat{A}_1 \\ \hat{E}_2 \end{bmatrix}^{*} \begin{bmatrix} \hat{A}_1 \\ \hat{E}_2 \end{bmatrix} - \begin{bmatrix} \hat{E}_1 \\ \hat{A}_2 \end{bmatrix}^{*} \begin{bmatrix} \hat{E}_1 \\ \hat{A}_2 \end{bmatrix} < 0 \tag{125}$$

or, equivalently

$$\bar{\sigma}\left(\begin{bmatrix} \hat{A}_1 \\ \hat{E}_2 \end{bmatrix} \begin{bmatrix} \hat{E}_1 \\ \hat{A}_2 \end{bmatrix}^{-1} \right) < 1. \tag{126}$$

Now

$$QT\tilde{\mathbf{\Delta}}(QT)^{-1} = Q\tilde{\mathbf{\Delta}}Q^{-1} =: \begin{bmatrix} \mathbf{\Delta}_1 & 0 \\ 0 & \mathbf{\Delta}_2 \end{bmatrix} \tag{127}$$

where $\mathbf{\Delta}_1$ and $\mathbf{\Delta}_2$ are diagonal operators, whose elements consist of the operators $\mathbf{S}_i^{-1}$; it thus follows that $\mathbf{\Delta}_2^{-1}$ exists, and that $\|\mathbf{\Delta}_1\|_{\ell_2} = \|\mathbf{\Delta}_2^{-1}\|_{\ell_2} = 1$. We have the following set of equalities:

$$\begin{aligned} & A_{\mathbf{SS}}^{-} - \tilde{\mathbf{\Delta}} A_{\mathbf{SS}}^{+} \\ &= (QT)^{-1}\left(\hat{E} - \begin{bmatrix} \mathbf{\Delta}_1 & 0 \\ 0 & \mathbf{\Delta}_2 \end{bmatrix} \hat{A}\right) QT \qquad (128) \\ &= (QT)^{-1} \begin{bmatrix} I & 0 \\ 0 & -\mathbf{\Delta}_2 \end{bmatrix} \left(I - \begin{bmatrix} \mathbf{\Delta}_1 & 0 \\ 0 & \mathbf{\Delta}_2^{-1} \end{bmatrix} \right. \\ & \qquad \times \left. \begin{bmatrix} \hat{A}_1 \\ \hat{E}_2 \end{bmatrix} \begin{bmatrix} \hat{E}_1 \\ \hat{A}_2 \end{bmatrix}^{-1} \right) \begin{bmatrix} \hat{E}_1 \\ \hat{A}_2 \end{bmatrix} QT. \qquad (129) \end{aligned}$$

Since $\mathbf{\Delta}_1$ and $\mathbf{\Delta}_2^{-1}$ are unitary operators, by the inequality in (126), we may express $(A_{\mathbf{SS}}^{-} - \tilde{\mathbf{\Delta}} A_{\mathbf{SS}}^{+})^{-1}$ as the following bounded operator (see [6, p. 169]):

$$\left(\begin{bmatrix} \hat{E}_1 \\ \hat{A}_2 \end{bmatrix} QT \right)^{-1} \left(\sum_{j=0}^{\infty} \left(\begin{bmatrix} \mathbf{\Delta}_1 & 0 \\ 0 & \mathbf{\Delta}_2^{-1} \end{bmatrix} \begin{bmatrix} \hat{A}_1 \\ \hat{E}_2 \end{bmatrix} \begin{bmatrix} \hat{E}_1 \\ \hat{A}_2 \end{bmatrix}^{-1} \right)^{j} \right) \times \begin{bmatrix} I & 0 \\ 0 & -\mathbf{\Delta}_2^{-1} \end{bmatrix} QT \tag{130}$$

as required. ■

Proposition 6: If $(A_{\mathbf{SS}}^{-} - \tilde{\mathbf{\Delta}} A_{\mathbf{SS}}^{+})$ is invertible on ℓ_2, then $(\mathbf{\Delta}_{\mathbf{S},\mathbf{m}} - A_{\mathbf{SS}})$ is invertible on ℓ_2.

Proof of Proposition 6: Define $\mathbf{\Delta}_{-} = \mathbf{diag}(I_{m_1}, -\mathbf{S}_1^{-1} I_{m_{-1}}, \ldots, -\mathbf{S}_L^{-1} I_{m_{-L}})$. Since $\mathbf{\Delta}_{\mathbf{S},\mathbf{m}} \mathbf{\Delta}_{\mathbf{S},\mathbf{m}}^{-1}, \mathbf{\Delta}_{-}, \mathbf{\Delta}_{-}^{-1}$, and $\tilde{\mathbf{\Delta}}$ are bounded operators on ℓ_2, the result follows from

$$\begin{aligned} \mathbf{\Delta}_{\mathbf{S},\mathbf{m}} - A_{\mathbf{SS}} &= \mathbf{\Delta}_{\mathbf{S},\mathbf{m}} \mathbf{\Delta}_{-}^{-1} \left(\mathbf{\Delta}_{-} - \mathbf{\Delta}_{-} \mathbf{\Delta}_{\mathbf{S},\mathbf{m}}^{-1} A_{\mathbf{SS}} \right) \\ &= \mathbf{\Delta}_{\mathbf{S},\mathbf{m}} \mathbf{\Delta}_{-}^{-1} \left(A_{\mathbf{SS}}^{-} - \tilde{\mathbf{\Delta}} A_{\mathbf{SS}}^{+} \right). \qquad (131) \end{aligned}$$

■

Stability: Now that we have shown that the system is well-posed, we can construct the state transition operator $\mathbf{A} = A_{\mathbf{TT}} + A_{\mathbf{TS}}(\mathbf{\Delta}_{\mathbf{S},\mathbf{m}} - A_{\mathbf{SS}})^{-1} A_{\mathbf{ST}}$. We have the following Lyapunov type of result.

Proposition 7: Let $x \in \ell_2$, and let $p = \mathbf{A}x$. If $J < 0$, then

$$\langle p, X_{\mathbf{T}} x \rangle_{\ell_2} + \langle X_{\mathbf{T}} x, p \rangle_{\ell_2} \le -\beta \|x\|_{\ell_2}^2 \tag{132}$$

for some positive constant β.

Proof of Proposition 7: Define $v = (\mathbf{\Delta}_{\mathbf{S},\mathbf{m}} - A_{\mathbf{SS}})^{-1} A_{\mathbf{ST}} x$, and $w = \mathbf{\Delta}_{\mathbf{S},\mathbf{m}} v = A_{\mathbf{ST}} x + A_{\mathbf{SS}} v$. Since J is strictly negative definite

$$\langle (x, v, 0), J(x, v, 0) \rangle_{\ell_2} \le -\beta \left(\|x\|_{\ell_2}^2 + \|v\|_{\ell_2}^2 \right) \le -\beta \|x\|_{\ell_2}^2 \tag{133}$$

for some strictly positive constant β. Define q_{+} and q_{-} in ℓ_2 as

$$q_{+} := [A_{\mathbf{ST}}^{+} \quad A_{\mathbf{SS}}^{+}](x, v) = (w_1, v_{-1}, w_2, v_{-2}, \ldots, v_{-L}) \tag{134}$$

$$q_{-} := [A_{\mathbf{ST}}^{-} \quad A_{\mathbf{SS}}^{-}](x, v) = (v_1, w_{-1}, v_2, w_{-2}, \ldots, w_{-L}). \tag{135}$$

It can readily be verified by direct substitution that

$$\begin{aligned} \langle (x, v, 0), J(x, v, 0) \rangle_{\ell_2} \ge{} & \langle p, X_{\mathbf{T}} x \rangle_{\ell_2} + \langle X_{\mathbf{T}} x, p \rangle_{\ell_2} \\ & + \langle q_{+}, X_{\mathbf{S}} q_{+} \rangle_{\ell_2} - \langle q_{-}, X_{\mathbf{S}} q_{-} \rangle_{\ell_2}. \qquad (136) \end{aligned}$$

Note, however, that

$$q_{+} = \mathbf{diag}\left(\mathbf{S}_1 I_{m_1}, I_{m_{-1}}, \mathbf{S}_2 I_{m_2}, I_{m_{-2}}, \ldots, I_{m_{-L}}\right) v \tag{137}$$

$$q_{-} = \mathbf{diag}\left(I_{m_1}, \mathbf{S}_1^{-1} I_{m_{-1}}, I_{m_2}, \mathbf{S}_2^{-1} I_{m_{-2}}, \ldots, \mathbf{S}_L^{-1} I_{m_{-L}}\right) v. \tag{138}$$

Thus, $q_{+} = \mathbf{\Delta}_{\mathbf{S},\hat{\mathbf{m}}} q_{-}$, where $\hat{\mathbf{m}} = (m_0, m_1 + m_{-1}, 0, \ldots, m_L + m_{-L}, 0)$. Also note that $\mathbf{\Delta}_{\mathbf{S},\hat{\mathbf{m}}}$ commutes with $X_{\mathbf{S}}$, and that $\mathbf{\Delta}_{\mathbf{S},\hat{\mathbf{m}}}^{*} \mathbf{\Delta}_{\mathbf{S},\hat{\mathbf{m}}} = \mathbf{I}$. Thus

$$\langle q_{+}, X_{\mathbf{S}} q_{+} \rangle_{\ell_2} = \langle q_{-}, \mathbf{\Delta}_{\mathbf{S},\hat{\mathbf{m}}}^{*} X_{\mathbf{S}} \mathbf{\Delta}_{\mathbf{S},\hat{\mathbf{m}}} q_{-} \rangle_{\ell_2} = \langle q_{-}, X_{\mathbf{S}} q_{-} \rangle_{\ell_2}. \tag{139}$$

This completes the proof. ■

The proof that $\exp(\mathbf{A}t)$ is exponentially stable now follows directly from the Lyapunov theorem [10, Th. 5.1.3].

Performance: We will next show that $\|\mathbf{M}\|_{\mathcal{L}_2} < 1$. Since the system is well-posed and stable, for any d in $\mathcal{L}_2$ there exist x, v, and z in $\mathcal{L}_2$ which satisfy (18) and (19), where $x(t = 0) = 0$. Since J is strictly negative

$$\langle (x, v, d), J(x, v, d) \rangle_{\mathcal{L}_2} \le -\beta \|d\|_{\mathcal{L}_2}^2 \tag{140}$$

for some strictly positive constant β. Let $w = \mathbf{\Delta}_{\mathbf{S},\mathbf{m}}v$. Define q_+ and q_- in $\mathcal{L}_2$ as

$$q_+ := [A^+_{\mathbf{ST}} \quad A^+_{\mathbf{SS}} \quad B^+_{\mathbf{S}}](x,v,d) = (w_1, v_{-1}, w_2, v_{-2}, \ldots, v_{-L}) \tag{141}$$

$$q_- := [A^-_{\mathbf{ST}} \quad A^-_{\mathbf{SS}} \quad B^-_{\mathbf{S}}](x,v,d) = (v_1, w_{-1}, v_2, w_{-2}, \ldots, w_{-L}). \tag{142}$$

It can readily be verified by expanding the inner product in (140) that

$$\langle \dot{x}, X_{\mathbf{T}}x\rangle_{\mathcal{L}_2} + \langle X_{\mathbf{T}}x, \dot{x}\rangle_{\mathcal{L}_2} + \langle q_+, X_{\mathbf{S}}q_+\rangle_{\mathcal{L}_2} - \langle q_-, X_{\mathbf{S}}q_-\rangle_{\mathcal{L}_2} + \|z\|^2_{\mathcal{L}_2} \le (1-\beta)\|d\|^2_{\mathcal{L}_2}. \tag{143}$$

As in the proof of stability, it can be shown that $\langle q_+(t), X_{\mathbf{S}}q_+(t)\rangle_{\ell_2} = \langle q_-(t), X_{\mathbf{S}}q_-(t)\rangle_{\ell_2}$ for all t, and thus $\langle q_+, X_{\mathbf{S}}q_+\rangle_{\mathcal{L}_2} - \langle q_-, X_{\mathbf{S}}q_-\rangle_{\mathcal{L}_2} = 0$. We will next show that $\langle \dot{x}, X_{\mathbf{T}}x\rangle_{\mathcal{L}_2} + \langle X_{\mathbf{T}}x, \dot{x}\rangle_{\mathcal{L}_2} = 0$, which will complete the proof

$$\langle \dot{x}, X_{\mathbf{T}}x\rangle_{\mathcal{L}_2} + \langle X_{\mathbf{T}}x, \dot{x}\rangle_{\mathcal{L}_2} = \int_0^\infty \left(\langle \dot{x}(t), X_{\mathbf{T}}x(t)\rangle_{\ell_2} + \langle X_{\mathbf{T}}x, \dot{x}(t)\rangle_{\ell_2}\right) dt \tag{144}$$

$$= \int_0^\infty \frac{d}{dt}\langle x(t), X_{\mathbf{T}}x(t)\rangle_{\ell_2}\, dt \tag{145}$$

$$= \langle x(\infty), X_{\mathbf{T}}x(\infty)\rangle_{\ell_2} - \langle x(0), X_{\mathbf{T}}x(0)\rangle_{\ell_2} = 0 \tag{146}$$

as required. ■

Proof of Theorem 2: We first state the following proposition; the proof follows from straightforward matrix manipulations, and is thus omitted.

Proposition 8: Given $\mathcal{M} = \{A, B, C, D, \mathbf{m}\}$, where $I - A_{\mathbf{SS}}$ is assumed to be invertible, define $\bar{\mathcal{M}} = \{\bar{A}, \bar{B}, \bar{C}, \bar{D}, \bar{\mathbf{m}}\} = f_{\mathrm{D2C}}(\mathcal{M})$. Consider the following set of equations:

$$\begin{bmatrix} p \\ w \\ z \end{bmatrix} = \begin{bmatrix} A_{\mathbf{TT}} & A_{\mathbf{TS}} & B_{\mathbf{T}} \\ A_{\mathbf{ST}} & A_{\mathbf{SS}} & B_{\mathbf{S}} \\ C_{\mathbf{T}} & C_{\mathbf{S}} & D \end{bmatrix} \begin{bmatrix} x \\ v \\ d \end{bmatrix}$$
$$\begin{bmatrix} \bar{v} \\ w \end{bmatrix} = \begin{bmatrix} H & -\sqrt{2}H \\ \sqrt{2} & -I \end{bmatrix} \begin{bmatrix} \bar{w} \\ v \end{bmatrix} \tag{147}$$

where H is defined in (44). Then, the following hold true.

1) For all $(x, v, d) \in \mathbb{R}^\bullet$, there exists $(p, w, z, \bar{w}, \bar{v}) \in \mathbb{R}^\bullet$ such that the equations in (147) are satisfied.
2) For all $(x, \bar{v}, d) \in \mathbb{R}^\bullet$, there exists $(p, w, z, \bar{w}, v) \in \mathbb{R}^\bullet$ such that the equations in (147) are satisfied, and furthermore

$$\begin{bmatrix} p \\ \bar{w} \\ z \end{bmatrix} = \begin{bmatrix} \bar{A}_{\mathbf{TT}} & \bar{A}_{\mathbf{TS}} & \bar{B}_{\mathbf{T}} \\ \bar{A}_{\mathbf{ST}} & \bar{A}_{\mathbf{SS}} & \bar{B}_{\mathbf{S}} \\ \bar{C}_{\mathbf{T}} & \bar{C}_{\mathbf{S}} & \bar{D} \end{bmatrix} \begin{bmatrix} x \\ \bar{v} \\ d \end{bmatrix}. \tag{148}$$

First, note that by the Schur complement formula the inequality in (51) is equivalent to

$$\bar{J} := \begin{bmatrix} \bar{A}^*X + X\bar{A} & X\bar{B} \\ \bar{B}^*X & -I \end{bmatrix} + [\bar{C} \quad \bar{D}]^*[\bar{C} \quad \bar{D}] < 0. \tag{149}$$

II $\Longrightarrow$ I:

Assume that there exists $0 \ne (x, v, d) \in \mathbb{R}^\bullet$, such that $(x, v, d)^*J(x, v, d) \ge 0$. If $I - A_{\mathbf{SS}}$ is invertible, define $\bar{\mathcal{M}} = f_{\mathrm{D2C}}(\mathcal{M})$. By Proposition 8, there exists $(p, w, z, \bar{w}, \bar{v}) \in \mathbb{R}^\bullet$ such that the equations in (147) are satisfied, where it can readily be verified that $(x, \bar{v}, d) \ne 0$. Upon substitution

$$(x,v,d)^*J(x,v,d) = p^*X_{\mathbf{T}}x + x^*X_{\mathbf{T}}p + z^*z - d^*d + q_+^*X_{\mathbf{S}}q_+ - q_-^*X_{\mathbf{S}}q_- \tag{150}$$

where q_+ and q_- are defined in (134) and (135). Similarly, it can readily be verified that

$$(x,\bar{v},d)^*\bar{J}(x,\bar{v},d) = p^*X_{\mathbf{T}}x + x^*X_{\mathbf{T}}p + z^*z - d^*d + \bar{w}^*X_{\mathbf{S}}\bar{v} + \bar{v}^*X_{\mathbf{S}}\bar{w}. \tag{151}$$

From (147), note that $w = \sqrt{2}\bar{w} - v$ and $\bar{v} = H\bar{w} - \sqrt{2}Hv$. Since $H^2 = I, \bar{w} = (1/\sqrt{2})(w+v)$ and $\bar{v} = (1/\sqrt{2})H(w-v)$. Note, however, that $w+v = q_+ + q_-$ and that $w - v = H(q_+ - q_-)$, and thus

$$\bar{w}^*X_{\mathbf{S}}\bar{v} + \bar{v}^*X_{\mathbf{S}}\bar{w} = \frac{1}{2}(q_+ + q_-)^*X_{\mathbf{S}}(q_+ - q_-) + \frac{1}{2}(q_+ - q_-)^*X_{\mathbf{S}}(q_+ + q_-) = q_+^*X_{\mathbf{S}}q_+ - q_-^*X_{\mathbf{S}}q_- \tag{152}$$

and, therefore, $(x,\bar{v},d)^*\bar{J}(x,\bar{v},d) = (x,v,d)^*J(x,v,d)$, which implies that $(x,\bar{v},d)^*\bar{J}(x,\bar{v},d) \ge 0$. Since $(x,\bar{v},d) \ne 0$, this completes the proof.

I $\Longrightarrow$ II:

If $(I - A_{\mathbf{SS}})$ is not invertible, assume that there exists $0 \ne v \in \mathbb{R}^\bullet$ such that $v = A_{\mathbf{SS}}v$. Note that $A^+_{\mathbf{SS}}v = v$ and that $A^-_{\mathbf{SS}}v = v$, and thus

$$(0,v,0)^*J(0,v,0) \ge v^*((A^+_{\mathbf{SS}})^*X_{\mathbf{S}}A^+_{\mathbf{SS}} - (A^-_{\mathbf{SS}})^*X_{\mathbf{S}}A^-_{\mathbf{SS}})v = 0. \tag{153}$$

If $(I - A_{\mathbf{SS}})$ is invertible, define $\bar{\mathcal{M}} = f_{\mathrm{D2C}}(\mathcal{M})$, and assume that there exists $0 \ne (x, \bar{v}, d) \in \mathbb{R}^\bullet$, such that $(x,\bar{v},d)^*\bar{J}(x,\bar{v},d) \ge 0$. By Proposition 8, there exists $(p, w, z, \bar{w}, v) \in \mathbb{R}^\bullet$ such that the equations in (147) and (148) are satisfied, where it can readily be verified that $(x, v, d) \ne 0$. As in the previous construction, it can also be readily verified that $(x,v,d)^*J(x,v,d) = (x,\bar{v},d)^*\bar{J}(x,\bar{v},d)$, which implies that $(x,v,d)^*J(x,v,d) \ge 0$. Since $(x,v,d) \ne 0$, this completes the proof. ■

Proof of Lemma 1: We will prove the result for two spatial dimensions ($\mathbf{s} = (s_1, s_2)$), where $s_1 \in \mathbb{Z}$ and $s_2 \in \{1, \ldots, N_2\}$; the general case is a straightforward extension of this special case.

Assume that $(I - A_{\mathbf{SS}})$ is not invertible. There thus exists real vector p such that $p^*p = 1, A_{\mathbf{SS}}p = p$. Partition p into $(p_1, p_{-1}, p_2, p_{-2})$, where $p_1 \in \mathbb{R}^{m_1}, p_{-1} \in \mathbb{R}^{m_{-1}}, p_2 \in \mathbb{R}^{m_2}, p_{-2} \in \mathbb{R}^{m_{-2}}$. For a fixed integer $N_1 > 1$, define signal $v \in \ell_2$ as follows:

$$v(s_1, s_2) = p, \quad \text{for } 1 \le s_1 \le N_1 \quad 1 \le s_2 \le N_2 \tag{154}$$

$$= 0, \quad \text{for all other } \mathbf{s}. \tag{155}$$

Let $n = (\mathbf{\Delta}_{\mathbf{S},\mathbf{m}} - A_{\mathbf{SS}})v$. It follows that

$$n(0, s_2) = (p_1, 0, 0, 0), \quad \text{for } 1 \le s_2 \le N_2 \tag{156}$$
$$n(1, s_2) = (0, -p_{-1}, 0, 0), \quad \text{for } 1 \le s_2 \le N_2 \tag{157}$$
$$n(N_1, s_2) = (-p_1, 0, 0, 0), \quad \text{for } 1 \le s_2 \le N_2 \tag{158}$$
$$n(N_1 + 1, s_2) = (0, p_{-1}, 0, 0), \quad \text{for } 1 \le s_2 \le N_2 \tag{159}$$
$$= 0, \quad \text{for all other } \mathbf{s}. \tag{160}$$

Note that $\|v\|_{\ell_2} = \sqrt{N_1 N_2}$ and that $\|n\|_{\ell_2} \le \sqrt{2N_2}$. Since N_1 is arbitrary, we can make $\|v\|_{\ell_2}/\|n\|_{\ell_2}$ as large as we want, which either proves that the inverse of $(\mathbf{\Delta}_{\mathbf{S},\mathbf{m}} - A_{\mathbf{SS}})$ is unbounded, or that there are multiple solutions to $n = (\mathbf{\Delta}_{\mathbf{S},\mathbf{m}} - A_{\mathbf{SS}})v$ for fixed n; either way, this demonstrates that $(\mathbf{\Delta}_{\mathbf{S},\mathbf{m}} - A_{\mathbf{SS}})$ does not have an inverse in ℓ_2, as required. ■

Proof of Proposition 3: Let R_2 and S_2 be full-column rank matrices in $\mathbb{R}^{n \times k}$ that solve the following equation:

$$R_2 S_2^* = I - R_1 S_1. \tag{161}$$

Note that we may express R_2 as $R_2 = (I - R_1 S_1) S_2 (S_2^* S_2)^{-1}$. Let the columns of $S_2^\perp$ in $\mathbb{R}^{n \times n-k}$ form a basis for the null space of S_2^*: $S_2^* S_2^\perp = 0$. We claim that the columns of $(S_2^\perp, -S_1 S_2^\perp)$ form a basis for the null space of the following two matrices:

$$\begin{bmatrix} S_1 & I \\ S_2^* & 0 \end{bmatrix} \quad \begin{bmatrix} I & R_1 \\ 0 & R_2^* \end{bmatrix}. \tag{162}$$

This follows by direct multiplication and from the following sequence of equalities:

$$\begin{aligned} -R_2^* S_1 S_2^\perp &= -(S_2^* S_2)^{-1} S_2^* S_1 (I - R_1 S_1) S_2^\perp \\ &= -(S_2^* S_2)^{-1} S_2^* S_1 R_2 S_2^* S_2^\perp = 0. \end{aligned} \tag{163}$$

Let $S := \begin{bmatrix} S_1 & S_2 \\ S_2^* & S_3 \end{bmatrix}$ be the unique solution to $\begin{bmatrix} S_1 & I \\ S_2^* & 0 \end{bmatrix} = S \begin{bmatrix} I & R_1 \\ 0 & R_2^* \end{bmatrix}$. Note that S_3 is symmetric

$$\begin{aligned} S_2^* R_1 + S_3 R_2^* = 0 &\Longrightarrow R_2 S_2^* R_1 + R_2 S_3 R_2^* = 0 \\ &\Longrightarrow (I - R_1 S_1) R_1 + R_2 S_3 R_2^* = 0 \\ &\Longrightarrow (I - R_1 S_1) R_1 + R_2 S_3^* R_2^* = 0 \\ &\Longrightarrow R_2 (S_3 - S_3^*) R_2^* = 0 \\ &\Longrightarrow S_3 = S_3^* \end{aligned}$$

which implies that S is symmetric. Similarly, let $R := \begin{bmatrix} R_1 & R_2 \\ R_2^* & R_3 \end{bmatrix}$ be the unique symmetric solution to

$$\begin{bmatrix} R_1 & I \\ R_2^* & 0 \end{bmatrix} = R \begin{bmatrix} I & S_1 \\ 0 & S_2^* \end{bmatrix}. \tag{164}$$

Left multiplying (164) by S, it can readily be verified from the equality

$$\begin{bmatrix} S_1 & S_2 \\ S_2^* & S_3 \end{bmatrix} \begin{bmatrix} R_1 & I \\ R_2^* & 0 \end{bmatrix} = SR \begin{bmatrix} I & S_1 \\ 0 & S_2^* \end{bmatrix} \tag{165}$$

that $SR = I$, as required. ■

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References

[1] P. Apkarian and P. Gahinet, "A convex characterization of gain-scheduled H-infinity controllers," *IEEE Trans. Automat. Contr.*, vol. 40, pp. 853–864, May 1995.

[2] G. Balas, J. Doyle, K. Glover, A. Packard, and R. Smith, *The μ Analysis and Synthesis Toolbox*. Natick, MA: The MathWorks Inc., 1991.

[3] B. Bamieh, F. Paganini, and M. Dahleh, "Distributed control of spatially invariant systems," *IEEE Trans. Automat. Contr.*, vol. 47, pp. 1091–1118, July 2002.

[4] S. P. Banks, *State-Space and Frequency Domain Methods in the Control of Distributed Parameter Systems*. New York: Peter Peregrinus, 1983.

[5] C. L. Beck and R. D'Andrea, "Simplification of spatially distributed systems," *Proc. IEEE Conf. Decision Control*, pp. 620–625, 1999.

[6] B. Bollobas, *Linear Analysis*. Cambridge, U.K.: Cambridge Univ. Press, 1990.

[7] S. P. Boyd, L. El. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA: SIAM, 1994. SIAM studies in applied mathematics.

[8] R. W. Brockett and J. L. Willems, "Discretized partial differential equations: Examples of control systems defined on modules," *Automatica*, vol. 10, pp. 507–515, 1974.

[9] D. F. Chichka and J. L. Speyer, "Solar-powered, formation-enhanced aerial vehicle system for sustained endurance," in *Proc. Amer. Control Conf.*, 1998, pp. 684–688.

[10] R. F. Curtain and H. J. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*. New York: Springer-Verlag, 1995.

[11] R. D'Andrea, "A linear matrix inequality approach to decentralized control of distributed parameter systems," in *Proc. Amer. Control Conf.*, 1998, pp. 1350–1354.

[12] ——, "Linear matrix inequalities, multidimensional system optimization, and control of spatially distributed systems: An example," in *Proc. Amer. Control Conf.*, 1999, pp. 2713–2717.

[13] ——, "Software for modeling, analysis, and control design for multidimensional systems," *Proc. IEEE Int. Symp. Computer-Aided Control System Design*, pp. 24–27, 1999.

[14] ——, "Extension of Parrott's theorem to nondefinite scalings," *IEEE Trans. Automat. Contr.*, vol. 45, pp. 937–940, May 2000.

[15] R. D'Andrea and C. Beck, "Temporal discretization of spatially distributed systems," *Proc. IEEE Conf. Decision Control*, pp. 197–202, 1999.

[16] R. D'Andrea and R. S. Chandra, "Control of discrete time interconnected systems," *Proc. IEEE Conf. Decision Control*, pp. 240–245, 2002.

[17] R. D'Andrea, C. Langbort, and R. Chandra, "A state space approach to control of interconnected systems," in *Mathematical Systems Theory in Biology, Communication, Computation and Finance*. ser. IMA Book Series, J. Rosenthal, Ed. New York: Springer-Verlag, 2003.

[18] J. C. Doyle, K. Glover, P. Khargonekar, and B. A. Francis, "State-space solutions to standard H2 and H-infinity control problems," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 831–847, Aug. 1989.

[19] G. Dullerud and R. D'Andrea, "Distributed control of inhomogeneous systems, with boundary conditions," *Proc. IEEE Conf. Decision .Control*, pp. 186–190, 1999.

[20] G. E. Dullerud and R. D'Andrea, "Distributed control of heterogeneous systems," *IEEE Trans. Automat. Contr.*, submitted for publication.

[21] G. E. Dullerud, R. D'Andrea, and S. Lall, "Control of spatially varying distributed systems," *Proc. IEEE Conf. Decision Control*, pp. 1889–1893, 1998.

[22] J. M. Fowler and R. D'Andrea, "Distributed control of close formation flight," *Proc. IEEE Conf. Decision Control*, pp. 2972–2977, 2002.

[23] ——, "A formation flight experiment for research in control of interconnected systems," *Control Syst. Mag.*, 2003, to be published.

[24] P. Gahinet and P. Apkarian, "A linear matrix inequality approach to H-∞ control," *Int. J. Robust Nonlinear Control*, vol. 4, pp. 421–448, 1994.

[25] P. Gahinet, A. Nemirovskii, A. Laub, and M. Chilali, *The LMI Control Toolbox*. Natick, MA: The MathWorks, Inc., 1995.

[26] D. Gorinevsky and G. Stein, "Structured uncertainty analysis of robust stability for spatially distributed systems," in *Proc. IEEE Conf. Decision Control*, 2000.
[27] J. Kulkarni, R. D'Andrea, and B. Brandl, "Application of distributed control techniques to the adaptive secondary mirror of cornell's large atacama telescope," presented at the SPIE Astronomical Telescopes Instrumentation Conf., 2002.
[28] C. Langbort and R. D'Andrea, "Imposing boundary conditions for spatially interconnected systems," presented at the Amer. Control Conf., 2003.
[29] J. M. Maciejowski, *Multivariable Feedback Design*. Reading, MA: Addison-Wesley, 1989.
[30] S. M. Melzer and B. C. Kuo, "Optimal regulation of systems described by a countably infinite number of objects," *Automatica*, vol. 7, pp. 359–366, 1971.
[31] M. Napoli and B. Bamieh, "Modeling and observed design for an array of electrostatically actuated microcantilevers," *Proc. IEEE Conf. Decision .Control*, pp. 4274–4279, 2001.
[32] A. Packard, "Gain scheduling via linear fractional transformations," *Syst. Control Lett.*, vol. 22, no. 2, pp. 79–92, 1994.
[33] A. Packard and J. C. Doyle, "The complex structured singular value," *Automatica*, vol. 29, no. 1, pp. 71–109, 1993.
[34] A. Packard, K. Zhou, P. Pandey, and G. Becker, "A collection of robust control problems leading to LMIs," *Proc. IEEE Conf. Decision Control*, pp. 1245–1250, 1991.
[35] F. Paganini, "A recursive information flow system for distributed control arrays," in *Proc. Amer. Control Conf.*, 1999, pp. 3821–3825.
[36] A. Rantzer, "On the Kalman–Yakubovich–Popov lemma," *Syst. Control Lett.*, vol. 28, no. 1, pp. 7–10, 1996.
[37] H. Raza and P. Ioannou, "Vehicle following control design for automated highway systems," *IEEE Control Syst.*, vol. 16, pp. 43–60, June 1996.
[38] M. G. Safonov and D. J. N. Limbeer, "Simplyfing the H-infinity theory via loop shifting," *Proc. IEEE Conf. Decision Control*, 1988.
[39] G. B. Shaw, D. W. Miller, and D. E. Hastings, "The generalized information network analysis methodology for distributed satellite systems," Ph.D. dissertation, Mass. Inst. Technol., Cambridge, MA, 1998.
[40] G. E. Stewart, "Two-dimensional loop shaping controller design for paper machine cross-directional processes," Ph.D. dissertation, Univ. British Columbia, Vancouver, BC, Canada, 2000.
[41] M. E. Taylor, *Partial Differential Equations I: Basic Theory*. New York: Springer-Verlag, 1996.
[42] J. D. Wolfe, D. F. Chichka, and J. L. Speyer, "Decentralized controllers for unmmaned aerial vehicle formation flight," *Amer. Inst. Aeronautics Astronautics*, 1996.
[43] F. Wu and E. Yildizoglu, "A distributed gain-scheduling control design for a flexible beam problem," in *Proc. Amer. Control Conf.*, 2002, pp. 3174–3179.
[44] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*. Upper Saddle River, NJ: Prentice-Hall, 1995.

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